Definitions: a The set $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\} \subseteq \mathbb{R}^n$ is called *orthogonal* if and only if all the vectors are non-zero and

$$\underline{v}_i \cdot \underline{v}_j = 0, \qquad i \neq j,\ i, j = 1 \dots p$$

(The vectors in an orthogonal set are mutually perpendicular to each other.)

b The set $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_p\} \subseteq \mathbb{R}^n$ is called *orthonormal* if and only if

$$\underline{u}_i \cdot \underline{u}_j = 0, \qquad i \neq j,\ i, j = 1 \dots p.$$
$$\underline{u}_i \cdot \underline{u}_i = 1,\ \ i = 1, 2, \dots p$$

So this is a set of mutually orthogonal vectors that are all unit vectors.

Remark: If $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\} \subseteq \mathbb{R}^n$ is an orthogonal set, then there is a corresponding orthonormal set

$$\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_p\} = \left\{ \frac{\underline{v}_1}{\|\underline{v}_1\|}, \frac{\underline{v}_2}{\|\underline{v}_2\|}, \dots, \frac{\underline{v}_p}{\|\underline{v}_p\|} \right\}$$

which spans the same subspace as the original orthogonal set.

Examples of ortho-normal sets you know already:

1) The standard basis $\{\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n\} \subseteq \mathbb{R}^n$, or any subset of the standard basis vectors.

2) Rotated bases in $\mathbb{R}^2$. $\{\underline{u}_1, \underline{u}_2\} = \left\{ \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \right\}.$

Theorem (why orthonormal sets are very good bases): Let $B = \{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_p\} \subseteq \mathbb{R}^n$ be orthonormal. Let $W = span\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_p\}$. Then

a) $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_p\}$ is linearly independent, so a basis for W.

Friday!

If $c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_p\vec{u}_p = \vec{0}$

dot with each $\vec{u}_j$: $c_1 \cancel{\vec{u}_1 \cdot \vec{u}_j}^{0} + c_2 \cancel{\vec{u}_2 \cdot \vec{u}_j}^{0} + \dots c_j\, \vec{u}_j \cdot \vec{u}_j + \dots + c_p \cancel{\vec{u}_p \cdot \vec{u}_j}^{0} = \vec{0}\cdot\vec{u}_j = 0$

$c_j = 0.$

b) For $\underline{w} \in W$, the coordinate vector $[\underline{w}]_B = \begin{bmatrix} \underline{u}_1 \cdot \underline{w} \\ \underline{u}_2 \cdot \underline{w} \\ \vdots \\ \underline{u}_p \cdot \underline{w} \end{bmatrix}$ is directly computable. In other words,

If $\vec{u}_j \cdot (c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_p\vec{u}_p) = \vec{w}\cdot\vec{u}_j$

$0 + 0 + \dots c_j + 0 \dots + 0 = \vec{w}\cdot\vec{u}_j$

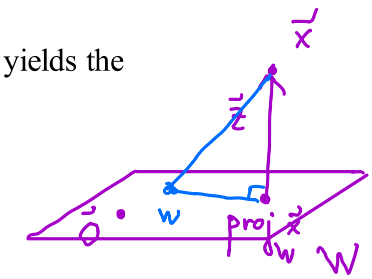
$$\underline{\boldsymbol{w}} = (\underline{\boldsymbol{u}}_1 \bullet \underline{\boldsymbol{w}})\underline{\boldsymbol{u}}_1 + (\underline{\boldsymbol{u}}_2 \bullet \underline{\boldsymbol{w}})\underline{\boldsymbol{u}}_2 + \ldots + (\underline{\boldsymbol{u}}_p \bullet \underline{\boldsymbol{w}})\underline{\boldsymbol{u}}_p$$

c) Let $\mathbf{x} \in \mathbb{R}^n$. Then there is a unique nearest point to $\mathbf{x}$ in W, which we call $proj_W \mathbf{x}$, ("the projection of $\mathbf{x}$ onto W.") The formula for this projection is given by

$$proj_W \mathbf{x} = (\mathbf{u}_1 \bullet \mathbf{x})\mathbf{u}_1 + (\mathbf{u}_2 \bullet \mathbf{x})\mathbf{u}_2 + \ldots + (\mathbf{u}_p \bullet \mathbf{x})\mathbf{u}_p .$$

(As should be the case, projection onto W leaves elements of W fixed.)

Proof: We will use the Pythagorean Theorem to show that the formula above for $proj_W \mathbf{x}$ yields the nearest point in W to $\mathbf{x}$:



Define

$$\mathbf{z} = \mathbf{x} - proj_W \mathbf{x} \quad \vec{z} \perp W$$

$$\vec{u}_j \bullet \mathbf{z} = \left(\mathbf{x} - (\mathbf{u}_1 \bullet \mathbf{x})\mathbf{u}_1 - (\mathbf{u}_2 \bullet \mathbf{x})\mathbf{u}_2 - \ldots - (\mathbf{u}_p \bullet \mathbf{x})\mathbf{u}_p .\right) \bullet \vec{u}_j$$

Then for $j = 1, 2, \ldots p$,

$$\mathbf{z} \bullet \mathbf{u}_j = \mathbf{x} \bullet \mathbf{u}_j - \mathbf{x} \bullet \mathbf{u}_j = 0 .$$

So $\mathbf{z} \perp W$, i.e.

$$\mathbf{z} \bullet (t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \ldots t_p \mathbf{u}_p) = 0$$

for all choices of the weight vector $\mathbf{t}$.

Let $\mathbf{w} \in W$. Then

$$\|\mathbf{x} - \mathbf{w}\|^2 = \|(\mathbf{x} - proj_W \mathbf{x}) + (proj_W \mathbf{x} - \mathbf{w})\|^2 .$$

Since $(\mathbf{x} - proj_W \mathbf{x}) = \mathbf{z}$ and since $(proj_W \mathbf{x} - \mathbf{w}) \in W$, we have the Pythagorean Theorem

$$\|\mathbf{x} - \mathbf{w}\|^2 = \|\mathbf{x} - proj_W \mathbf{x}\|^2 + \|proj_W \mathbf{x} - \mathbf{w}\|^2$$

$$\|\mathbf{x} - \mathbf{w}\|^2 = \|\mathbf{z}\|^2 + \|proj_W \mathbf{x} - \mathbf{w}\|^2 .$$

So $\|\mathbf{x} - \mathbf{w}\|^2$ is always at least $\|\mathbf{z}\|^2$, with equality if and only if $\mathbf{w} = proj_W \mathbf{x}$.

QED

Exercise 1

1a) Check that the set

$$B = \left\{ \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

is an orthonormal basis for $\mathbb{R}^3$.

we did this

1b) For $\underline{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ find the coordinate vector $[\underline{\boldsymbol{x}}]_B$ and check your answer.

& this

solution $[\underline{\boldsymbol{x}}]_B = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Exercise 2 Consider the plane from Tuesday

$$W = span\left\{\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\-2\end{bmatrix}\right\}$$

$W^{\perp}$: $Nul\begin{bmatrix}1 & 1 & 3\\ 1 & 0 & -2\end{bmatrix}$

$W^{\perp} = span\{[2\ -5\ 1]\}$ ✓

$\begin{bmatrix}1 & 1 & 3\\ 1 & 0 & -2\end{bmatrix}\begin{bmatrix}2\\-5\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

which is also given implicitly as a nullspace,

$$W = Nul\begin{bmatrix}2 & -5 & 1\end{bmatrix}.$$

or $Nul[2\ -5\ 1]$ is 2-dim'l.

$[2\ -5\ 1]\begin{bmatrix}1\\1\\3\end{bmatrix} = 0$

$[2\ -5\ 1]\begin{bmatrix}1\\0\\-2\end{bmatrix} = 0$

so $\left\{\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\-2\end{bmatrix}\right\}$ are a basis for W

2a) Verify that

basis. (in W)

$$B = \left\{\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\-2\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\1\end{bmatrix}\right\}$$

is an ortho-normal basis for W.

✓ $[2\ -5\ 1]\begin{bmatrix}2\\1\\1\end{bmatrix} = 0$

$\vec{b}_1 \cdot \vec{b}_2 = 0$

$\vec{b}_1 \cdot \vec{b}_1 = \frac{1}{5}\cdot 5 = 1$ ✓

$\vec{b}_2 \cdot \vec{b}_2 = \frac{1}{6}\cdot 6 = 1$

2b) Find $proj_W \mathbf{x}$ for $\mathbf{x} = \begin{bmatrix}7\\-3\\1\end{bmatrix}$. Then verify that $\mathbf{z} = \mathbf{x} - proj_W \mathbf{x}$ is perpendicular to W.

solution $proj_W \mathbf{x} = \begin{bmatrix}5\\2\\0\end{bmatrix}$

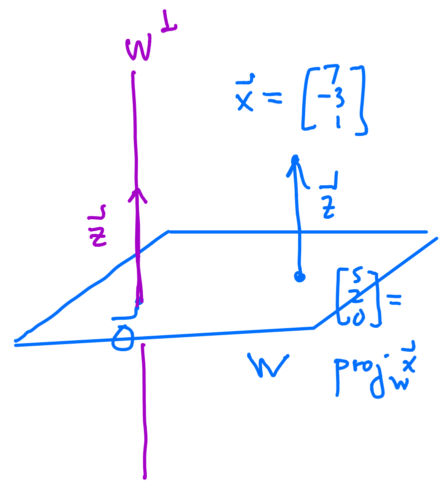
$$proj_W \begin{bmatrix}7\\-3\\1\end{bmatrix} = (\vec{x}\cdot\vec{b}_1)\vec{b}_1 + (\vec{x}\cdot\vec{b}_2)\vec{b}_2$$

$$= \underbrace{\left(\begin{bmatrix}7\\-3\\1\end{bmatrix}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\-2\end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\-2\end{bmatrix}}_{\frac{1}{5}5\begin{bmatrix}1\\0\\-2\end{bmatrix}} + \underbrace{\left(\begin{bmatrix}7\\-3\\1\end{bmatrix}\cdot\frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\1\end{bmatrix}\right)\frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\1\end{bmatrix}}_{\frac{1}{6}12\begin{bmatrix}2\\1\\1\end{bmatrix}}$$

$$= \begin{bmatrix}1\\0\\-2\end{bmatrix} + \begin{bmatrix}4\\2\\2\end{bmatrix} = \begin{bmatrix}5\\2\\0\end{bmatrix}$$

$\vec{z} = \vec{x} - proj_W \vec{x} \perp W$?

$$= \begin{bmatrix}7\\-3\\1\end{bmatrix} - \begin{bmatrix}5\\2\\0\end{bmatrix} = \begin{bmatrix}2\\-5\\1\end{bmatrix} \in W^{\perp}$$



Remark: As we mentioned, if $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p\} \subseteq \mathbb{R}^n$ is an *orthogonal* set (of mutually perpendicular vectors), then there is the corresponding orthonormal basis obtained from that set by normalizing, namely

$$\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_p\} = \left\{ \frac{\underline{v}_1}{\|\underline{v}_1\|}, \frac{\underline{v}_2}{\|\underline{v}_2\|}, \ldots, \frac{\underline{v}_p}{\|\underline{v}_p\|} \right\} \subseteq \mathbb{R}^n .$$

One can avoid square roots if one uses the original orthogonal basis rather than the ortho-normal one. This is the approach the text prefers. For example, for orthogonal bases, the very good basis theorem reads

Theorem (why orthogonal bases are very good bases): Let $B = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p\} \subseteq \mathbb{R}^n$ be orthogonal. Let $W = span\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p\}$. Then

a) $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p\}$ is linearly independent, so a basis for W.

b) For $\underline{w} \in W$,

$$\underline{w} = (\underline{u}_1 \cdot \underline{w})\underline{u}_1 + (\underline{u}_2 \cdot \underline{w})\underline{u}_2 + \ldots + (\underline{u}_p \cdot \underline{w})\underline{u}_p$$

$$\underline{w} = \frac{(\underline{v}_1 \cdot \underline{w})}{\|\underline{v}_1\|^2}\underline{v}_1 + \frac{(\underline{v}_2 \cdot \underline{w})}{\|\underline{v}_2\|^2}\underline{v}_2 + \ldots + \frac{(\underline{v}_p \cdot \underline{w})}{\|\underline{v}_p\|^2}\underline{v}_p$$

c) Let $\underline{x} \in \mathbb{R}^n$. Then there is a unique nearest point to $\underline{x}$ in W, which we call $proj_W \underline{x}$, ("the projection of $\underline{x}$ onto W.") The formula for this projection is given by

$$proj_W \underline{x} = (\underline{u}_1 \cdot \underline{x})\underline{u}_1 + (\underline{u}_2 \cdot \underline{x})\underline{u}_2 + \ldots + (\underline{u}_p \cdot \underline{x})\underline{u}_p .$$

$$proj_W \underline{x} = \frac{(\underline{v}_1 \cdot \underline{x})}{\|\underline{v}_1\|^2}\underline{v}_1 + \frac{(\underline{v}_2 \cdot \underline{x})}{\|\underline{v}_2\|^2}\underline{v}_2 + \ldots + \frac{(\underline{v}_p \cdot \underline{x})}{\|\underline{v}_p\|^2}\underline{v}_p .$$

You can see how that would have played out in the previous exercise.

Nov 16 (wikipedia)

Fri ~~Apr 6~~

- 6.3-6.4 Gram-Schmidt process for constructing ortho-normal (or orthogonal) bases. The $A = Q\,R$ matrix factorization. (I'll bring notes to class for the second topic, ~~if it looks like we'll have time~~ on Friday. Otherwise we'll discuss it on Monday.)

Announcements: Wed, part of today's notes

(a) checks all dot products at once $= I$

$$\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ -2 & 1 & 2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Warm-up Exercise: a) Check that $\beta = \left\{ \underset{\vec{b}_1}{\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}, \underset{\vec{b}_2}{\frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}, \underset{\vec{b}_3}{\frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}} \right\}$ is an orthonormal basis for $\mathbb{R}^3$

orthogonal / unit length

b) for $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, how would you have found $[\vec{x}]_\beta$ before today?

b) Old way: solve $\frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ for $\vec{c} = [\vec{x}]_\beta$

If basis is orthonormal, New way to find coords:

$$\left(c_1 \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + c_2 \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + c_3 \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}\right) \cdot \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

dot both sides with $\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

dotting with 1^{st} basis vector yields c_1

$$c_1 \cdot 1 + 0 + 0 = \frac{1}{3} 9 = 3 = c_1$$

dot with $\vec{b}_2$: $\frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

$$0 + c_2 + 0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 1 \qquad c_2 = 1$$

dot with $\vec{b}_3$:

$$0 + 0 + c_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = 2 \qquad c_3 = 2$$

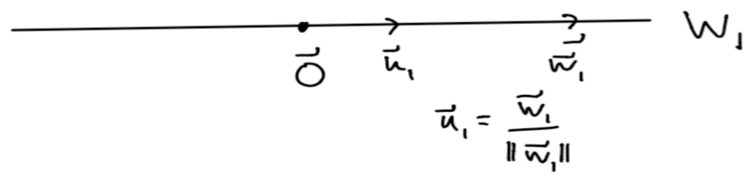
$$[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$ (you can check).

a) you could check lin. ind. same way.

(analogous for orthogonal vectors)

Start with a non-orthogonal basis $B = \{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_p\}$ for a subspace W of $\mathbb{R}^n$. How can you convert it into an orthonormal basis? Here's how! The inductive process is called *Gram-Schmidt orthogonalization*.

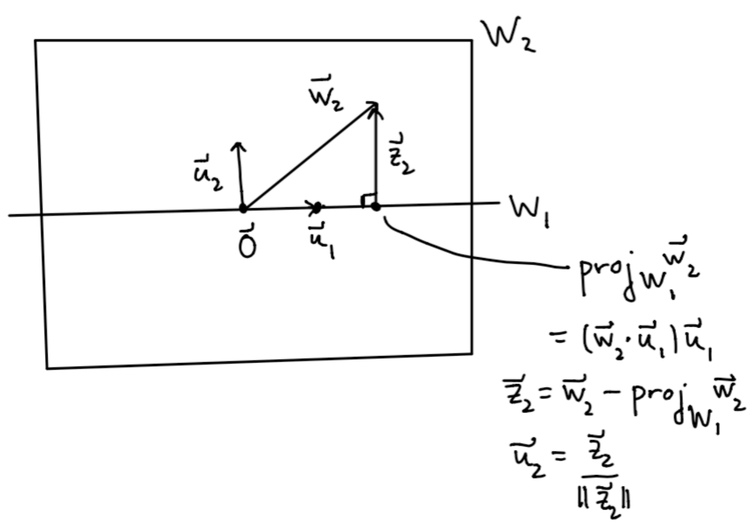
Let $W_1 = span\{\underline{\boldsymbol{w}}_1\}$. Define $\underline{\boldsymbol{u}}_1 = \dfrac{\underline{\boldsymbol{w}}_1}{\|\underline{\boldsymbol{w}}_1\|}$. Then $\{\underline{\boldsymbol{u}}_1\}$ is an orthonormal basis for W_1.



Let $W_2 = span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2\}$.

Let $\underline{\boldsymbol{z}}_2 = \underline{\boldsymbol{w}}_2 - proj_{W_1}\underline{\boldsymbol{w}}_2$, so $\underline{\boldsymbol{z}}_2 \perp \underline{\boldsymbol{u}}_1$.

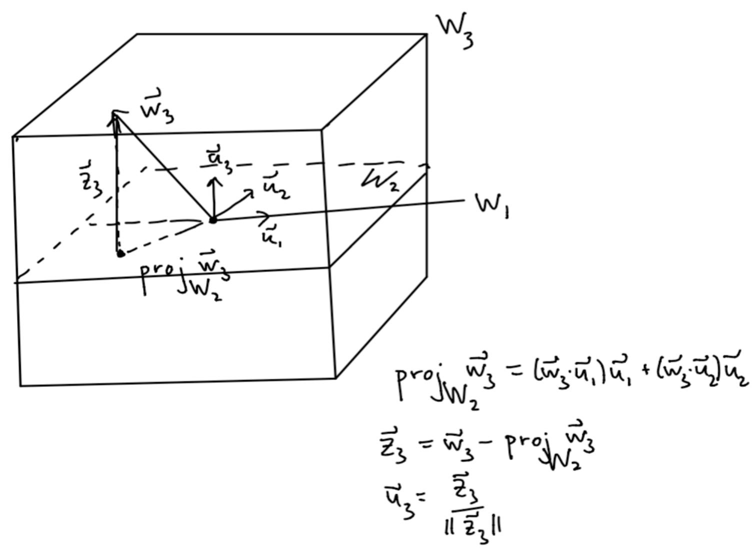
Define $\underline{\boldsymbol{u}}_2 = \dfrac{\underline{\boldsymbol{z}}_2}{\|\underline{\boldsymbol{z}}_2\|}$. So $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2\}$ is an orthonormal basis for W_2.



Let $W_3 = span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \underline{\boldsymbol{w}}_3\}$.

Let $\underline{\boldsymbol{z}}_3 = \underline{\boldsymbol{w}}_3 - proj_{W_2}\underline{\boldsymbol{w}}_3$, so $\underline{\boldsymbol{z}}_3 \perp W_2$.

Define $\underline{\boldsymbol{u}}_3 = \dfrac{\underline{\boldsymbol{z}}_3}{\|\underline{\boldsymbol{z}}_3\|}$. Then $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \underline{\boldsymbol{u}}_3\}$ is an orthonormal basis for W_3.



Inductively,

Let $W_j = span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_j\} = span\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ldots \underline{\boldsymbol{u}}_{j-1}, \underline{\boldsymbol{w}}_j\}$.

Let $\underline{\boldsymbol{z}}_j = \underline{\boldsymbol{w}}_j - proj_{W_{j-1}}\underline{\boldsymbol{w}}_j = \underline{\boldsymbol{w}}_j - (\underline{\boldsymbol{w}}_j \cdot \underline{\boldsymbol{u}}_1)\underline{\boldsymbol{u}}_1 - (\underline{\boldsymbol{w}}_j \cdot \underline{\boldsymbol{u}}_2)\underline{\boldsymbol{u}}_2 - \ldots (\underline{\boldsymbol{w}}_j \cdot \underline{\boldsymbol{u}}_{j-1})\underline{\boldsymbol{u}}_{j-1}$.

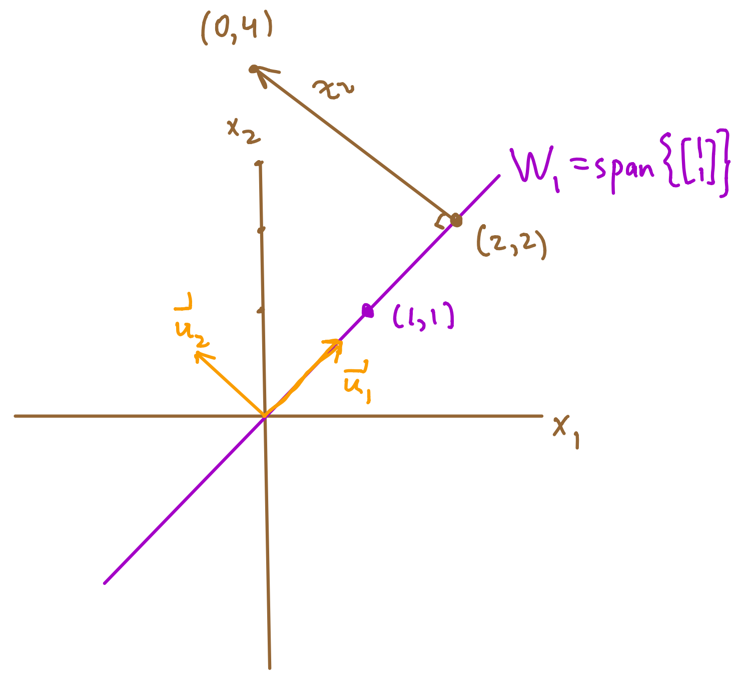
Define $\underline{\boldsymbol{u}}_j = \dfrac{\underline{\boldsymbol{z}}_j}{\|\underline{\boldsymbol{z}}_j\|}$. Then $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ldots \underline{\boldsymbol{u}}_j\}$ is an orthonormal basis for W_j.

Continue up to $j = p$.

Exercise 1 Perform Gram-Schmidt orthogonalization on the basis

$$B = \left\{ \underset{\vec{w}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{\vec{w}_2}{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} \right\}.$$

Sketch what you're doing, as you do it.



$$\vec{u}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{proj}_{W_1}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \underbrace{\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\frac{1}{\sqrt{2}}}_{\frac{1}{2}4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\vec{z}_2 = \vec{w}_2 - \text{proj}_{W_1}\vec{w}_2$$

$$= \begin{bmatrix} 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\vec{u}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

(if we'd reordered our set into $\left\{ \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ G.S. would yield $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$)