2h) Refer to the same diagram as in 2g, which is an $\mathbb{R}^n$ picture. Using the Pythagorean triangle with edges $(\underline{x} \cdot \underline{u})\underline{u}$, $\underline{z}$, $\underline{x}$ we have

$$\| (\underline{x} \cdot \underline{u})\underline{u} \|^2 + \|\underline{z}\|^2 = \|\underline{x}\|^2, \text{ i.e.}$$

• from $\vec{z} \perp (\vec{x}\cdot\vec{u})\vec{u}$
so

$$(\underline{x} \cdot \underline{u})^2 + \| \underline{z} \|^2 = \|\underline{x}\|^2.$$

The quantity $\underline{x} \cdot \underline{u}$ is called *the component of* $\underline{x}$ in the direction of $\underline{u}$, and from the formula above,

$$-\|x\| \leq \underline{x} \cdot \underline{u} \leq \| x\| .$$

(extreme cases are when $\|\vec{z}\| = 0$
i.e. $\vec{x}$ is already on the line

Define the angle θ between $\underline{v}$ and $\underline{w}$ the same way we would in $\mathbb{R}^2$, using the congruent triangle in the figure below, namely

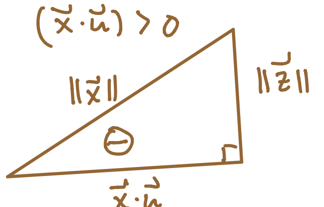
$$\cos(\theta) = \frac{(\underline{x} \cdot \underline{u})}{\|\underline{x}\|}.$$

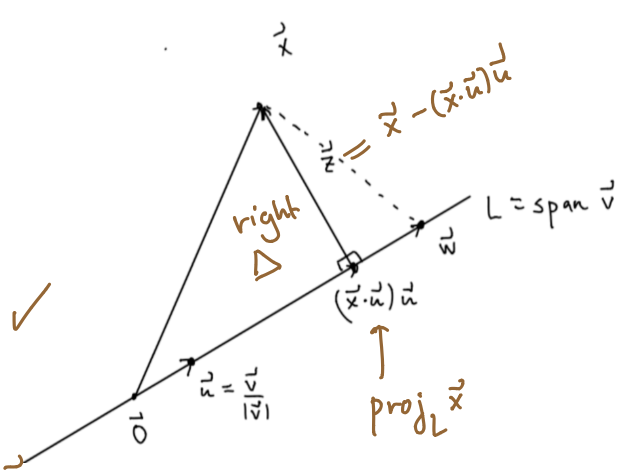
Notice that $-1 \leq \cos(\theta) \leq 1$ and so there is a unique θ with $0 \leq \theta \leq \pi$ for which the cos θ equation can hold. Substituting $\underline{u} = \dfrac{\underline{v}}{\|\underline{v}\|}$ gives the familiar formulas that you learned in multivariable Calculus for $\mathbb{R}^2$, $\mathbb{R}^3$, which now holds in $\mathbb{R}^n$.

$$\cos(\theta) = \frac{\left(\underline{x} \cdot \dfrac{\underline{v}}{\|\underline{v}\|}\right)}{\|\underline{x}\|} = \frac{(\underline{x} \cdot \underline{v})}{\|\underline{x}\| \, \|\underline{v}\|}, \text{ i.e.}$$

$$(\underline{x} \cdot \underline{v}) = \|\underline{x}\| \, \|\underline{v}\| \cos(\theta)$$

congruent Δ in $\mathbb{R}^2$:
if $(\vec{x}\cdot\vec{u}) > 0$

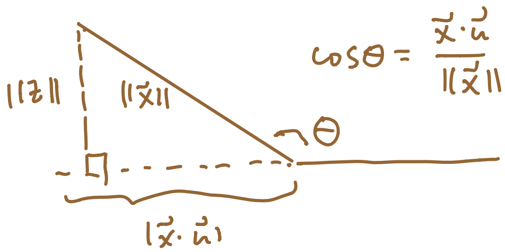




$$0 < \cos\theta = \frac{\vec{x}\cdot\vec{u}}{\|\vec{x}\|} = \frac{\vec{x}\cdot\frac{\vec{v}}{\|\vec{v}\|}}{\|\vec{x}\|} = \frac{\vec{x}\cdot\vec{v}}{\|x\|\,\|\vec{v}\|} \checkmark$$

if $\vec{x}\cdot\vec{u} < 0$

$$\cos\theta = \frac{\vec{x}\cdot\vec{u}}{\|\vec{x}\|} = \frac{\vec{x}\cdot\vec{v}}{\|\vec{x}\|\,\|\vec{v}\|} < 0$$



Tues Nov 13

- 6.2-6.3 Orthogonal complements, and the four fundamental subspaces of a matrix revisited.

Announcements:
- midterms returned tomorrow
- short HW due tomorrow
  (quiz will involve projection ideas)
- Today: $\cos\theta$ formula in $\mathbb{R}^n$ (Monday notes)
  then Tues notes

'til 12:58, so you can at least write down the problem

Warm-up Exercise: Let $L = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $\quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a) Compute $\operatorname{proj}_L \vec{x}$ for $\vec{x} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, using $\operatorname{proj}_L \vec{x} = (\vec{x}\cdot\vec{u})\vec{u}$ $\left(= \frac{(\vec{x}\cdot\vec{v})}{\|\vec{v}\|^2}\vec{v} \right)$

$\left(\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} \right)$

b) Find the angle $\theta = \measuredangle \vec{x},\vec{v}$ $= \frac{3}{4}\pi$

using $\cos\theta = \frac{\vec{x}\cdot\vec{v}}{\|\vec{x}\|\,\|\vec{v}\|}$

c) illustrate

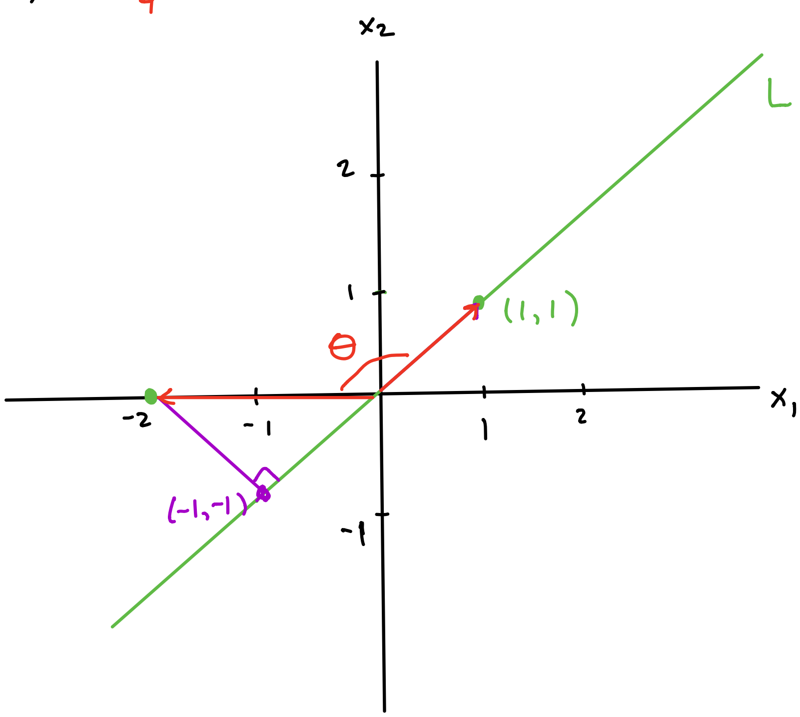
a) $\vec{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \left(\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} \right)$

$(\vec{x}\cdot\vec{u})\vec{u} = \left(\begin{bmatrix} -2 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \frac{1}{2}\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \frac{1}{2}(-2)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ ✓



b) $\cos\theta = \frac{\begin{bmatrix} -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\|\vec{x}\|\,\|\vec{v}\|} = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$

$\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3}{4}\pi$ ✓

Orthogonal complements, and the four subspaces associated with a matrix transformation, revisited more carefully than our first time through.

Let $W \subseteq \mathbb{R}^n$ be a subspace of dimension $1 \le p \le n$. The *orthogonal complement to W* is the collection of all vectors perpendicular to every vector in W. We write the orthogonal complement to W as $W^\perp$, and say "W perp". Let $B = \{\underline{w}_1, \underline{w}_2, \ldots \underline{w}_p\}$ be a basis for W. Let $\underline{v} \in W^\perp$. This means

$$\left(c_1 \underline{w}_1 + c_2 \underline{w}_2 + \ldots + c_p \underline{w}_p\right) \cdot \underline{v} = 0$$

for all linear combinations of the spanning vectors. Since the dot product distributes over linear combinations, the identity above expands as

$$c_1\left(\underline{w}_1 \cdot \underline{v}\right) + c_2\left(\underline{w}_2 \cdot \underline{v}\right) + \ldots + c_p\left(\underline{w}_p \cdot \underline{v}\right) = 0$$

for all possible weights. This is always true as soon as we check the special cases

$$\underline{w}_1 \cdot \underline{v} = \underline{w}_2 \cdot \underline{v} = \ldots = \underline{w}_p \cdot \underline{v} = 0.$$

In other words, $\underline{v} \in Nul\,A$ where A is the $m \times n$ matrix having the spanning vectors as rows:

$$A\,\underline{v} = \begin{bmatrix} \underline{w}_1^T \\ \underline{w}_2^T \\ \vdots \\ \underline{w}_p^T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \underline{\mathbf{0}}.$$

So

$$W^\perp = Nul\,A.$$

<u>Exercise 1</u> Find $W^\perp$ for $W = span\left\{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right\}$. $\subset \mathbb{R}^3$ W is a plane thru $\vec{0}$.

$\vec{x} \in W^\perp$ need $\vec{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 0$

$\vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 0$

$W^\perp = Nul \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_1 \\ R_1 \to R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right]$$

$$-R_1 + R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right]$$

$\dim W^\perp = 1$ (1 non-pivot).

$W^\perp = span\left\{\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}\right\}$

Theorem: Let A be any $m \times n$ matrix. Then $(Row\ A)^{\perp} = Nul\ A$.

$\vec{x} \in (\text{Row } A)^{\perp}$

$\Longleftrightarrow$ $\vec{x} \perp$ each row of A (because $\text{Row } A$ = span of rows of A)

$\Longleftrightarrow$ $A\vec{x} = \vec{0}$

Theorem. Conversely, let A be an $m \times n$ matrix. Then $(Nul\ A)^{\perp} = Row\ A$.

Let rank $A = r$ = # pivots = dim Row A. (& dim Col A)

dim Nul $A = n-r$ = # non-pivot columns

Let $\{\vec{z}_1, \vec{z}_2, \ldots \vec{z}_{n-r}\}$ be a basis for Nul A

So, $\vec{x} \in (\text{Nul } A)^{\perp}$ iff $\begin{bmatrix} \vec{z}_1^{\,T} \\ \vec{z}_2^{\,T} \\ \vdots \\ \vec{z}_{n-r}^{\,T} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

rank "Z" $= n-r$ (dim of row space = dim Nul A), so r non-pivot columns

• dim $(\text{Nul } A)^{\perp} = r$

• Row $A \subset (\text{Nul } A)^{\perp}$

because $Z\left[\text{row}_i(A)\right] = \vec{0}$.

dim Row $A = r$ as well so it has to be all of $(\text{Nul } A)^{\perp}$ by vector space theory.

Corollary Let $W \subseteq \mathbb{R}^n$ be a subspace. Then $(W^{\perp})^{\perp} = W$.

just use the two thms above.

Start with

$A = \begin{bmatrix} \vec{w}_1^{\,T} \\ \vec{w}_2^{\,T} \\ \vdots \\ \vec{w}_r^{\,T} \end{bmatrix}$

$\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_r\}$ basis for W.

$W^{\perp} = \text{Nul } A$ (1st Thm).

$(W^{\perp})^{\perp} = (\text{Nul } A)^{\perp}$

$= \text{Row } A$ (2nd Thm)

$= \text{span}\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_r\}$

$= W$

Exercise 2 For $W = span\left\{\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\-2\end{bmatrix}\right\}$ as in Exercise 1, verify that $(W^{\perp})^{\perp} = W$.

$$W^{\perp} = \text{span}\left\{\begin{bmatrix}2\\-5\\1\end{bmatrix}\right\}.$$

$$(W^{\perp})^{\perp} = \underbrace{\text{Nul}\begin{bmatrix}2 & -5 & 1\end{bmatrix}}_{}$$

dim = 2 (2 non-pivot cols).

$$\begin{bmatrix}2 & -5 & 1\end{bmatrix}\begin{bmatrix}1\\1\\3\end{bmatrix} = 0$$

$$\begin{bmatrix}2 & -5 & 1\end{bmatrix}\begin{bmatrix}1\\0\\-2\end{bmatrix} = 0$$

so $\left\{\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\-2\end{bmatrix}\right\}$ are a basis for $(W^{\perp})^{\perp} = W$