Math 2270-002
Homework due October 24.

Recall that problems which are not underlined are good for seeing if you can work with the underlying concepts; only the underlined problems need to be handed in. The Wednesday quiz will be drawn from all of these concepts and from these or related problems.

Remember that you are allowed (or even encouraged) to use technology in order to compute the reduced row echelon form of any matrix with more than 12 entries.

4.4 Coordinate systems
1, **3**, **5**, **7**, **9**, **11**, **13**, **15**, **21**, **27**, **31**

4.*5: Dimension of vector spaces*
3, **7**, **9**, **11**, 13, **19**, **25**, **27**, **28** (hint: use 27). *In 27, 28, we say that a vector space is infinite dimensional if there is not basis consisting of a finite number of vectors.*

4.6: rank, and the four fundamental subspaces of a matrix.
1, **5**, **7**, **9**, **11**, **17**, 19, 21, **27.**

....

Section 4.9 homework postponed until next assignment, due October 31
4.9: applications to Markov Chains
1, **2**, 11, **12**, **16**, **17**.

....

w8.1) (relates to section 4.6) In last week's homework you considered the matrix $A_{3 \times 6}$ given by

$$A := \begin{bmatrix} 1 & 2 & -1 & 2 & -1 \\ 2 & 4 & 0 & 6 & 2 \\ 1 & 2 & 2 & 5 & 5 \end{bmatrix}.$$

And found bases for *Col A* and *Nul A*. You used the reduced row echelon form of *A*, which is given by

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

w8.1a) Find a basis for *Row A*, the subspace of $\mathbb{R}^5$ spanned by the three rows of *A*. Hint: There is a very good basis available in one of the matrices above. What is the dimension of this subspace of $\mathbb{R}^5$?

w8.1b) Explain why the dimension of *Row B* always equals the dimension of *Col B*, for any matrix. Hint: Look at the what the reduced row echelon form of *B* tells you about these dimensions. The common value is called the *rank* of the matrix *B*.

Remark: Last week you understood the "rank plus nullity" theorem, namely that

$$dim\,(Nul\,B)\;+\;dim\,(Col\,B) = n$$

Notice that because the transpose operation interchanges rows and columns, $Row\ B$ is actually $Col(B^T)$. So applying the rank plus nullity theorem to the transpose matrix, we see that

$$dim\left(Nul\,B^T\right)\;+\;dim\left(Col\,B^T\right) = m.$$
$$dim\left(Nul\,B^T\right)\;+\;dim\,(Row\,B) = m.$$

w8.1c) In our example, A is a 3 × 5 matrix and A^T is 5 × 3. Find a basis for $Nul(A^T)$ and verify that in your example

$$dim\left(Nul\,A^T\right)\;+\;dim\,(Row\,A) = 3.$$

(Hint: when you column reduced A to find a very good basis for $Col\ A$, you were doing the same computations as you need to do to row reduce A^T.

w8.2) (Relates to section 4.6) Use the two dimension equalities from w8.1c above to explain why it is true that for any $m \times n$ matrix B, that the matrix equation

$$B\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$$

is consistent for all $\underline{\boldsymbol{b}} \in \mathbb{R}^m$ if and only if the homogeneous equation

$$B^T\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

has only the trivial solution $\underline{\mathbf{x}} = \underline{\mathbf{0}}$. (This is a generalization of a fact we know about square matrices, to general rectangular ones, because in the case that B is $n \times n$ these are two of the equivalent statements on our the very long list of things equivalent to B having an inverse matrix.)

w8.3) (Relates to section 4.4) Consider the set of points in the $x_1 - x_2$ plane that satisfy the equation $x_1 x_2 = 1$, i.e. the graph $x_2 = \frac{1}{x_1}$. This is a basic curve that you've studied since the time of precalculus. When you learned about conic sections it's possible someone told you that the resulting curve is a hyperbola, even though the equation is not in the standard form for a hypberbola that opens along one of the axes, e.g.

$$\frac{y_1^2}{a^2} - \frac{y_2^2}{b^2} = 1$$

for a hyperbola opening along a y_1 - axis.

The reason that the equation $x_1 x_2 = 1$ is not in the standard form for a hyperbola is because we are using the wrong coordinate system to describe the curve. Consider the rotated basis:

$$\beta = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\} = \{\underline{\boldsymbol{b}}_1, \underline{\boldsymbol{b}}_2\}$$

which is a rotation by an angle $\frac{\pi}{4}$ radians of the standard basis $\{\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2\}$. Show that if $\underline{\boldsymbol{x}} = [\underline{\boldsymbol{x}}]_E$ satisfies $x_1 x_2 = 1$ then the coordinates of $\underline{\boldsymbol{x}}$ with respect to β, $[\underline{\boldsymbol{x}}]_\beta = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, satisfy the hyperbola equation

$$\frac{y_1^2}{2} - \frac{y_2^2}{2} = 1$$

picture to motivate you: