Math 2270-002
Homework due October 17.

Recall that problems which are not underlined are good for seeing if you can work with the underlying concepts; only the underlined problems need to be handed in. The Wednesday quiz will be drawn from all of these concepts and from these or related problems.

Remember that you are allowed (or even encouraged) to use technology in order to compute the reduced row echelon form of any matrix with more than 12 entries.

4.1 Vector spaces and subspaces
1, **3**, **5**, **7**, 9, **11**, **13**, **19**, 21, **23**, **24**, **31**.

4.2: Null spaces, column spaces and linear transformations
1, **3**, **7**, **9**, **15**, **17**, 21, **25**, 27, **31**, **33**

4.3: Linear independent sets; bases
1, **3**, **5**, 9, 13, **15**, **19**, **21**, **31**, **33** *Use technology to reduce large matrices!*

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4.4 Coordinate systems
1, **3**, **5**, **7**, **9**, **11**, **13**, **15**, **21**, **27**, **31**

NOTE: The section 4.4 problems above are postponed until next week.
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w7.1) (parts a,b are just like 4.3.13) Consider the matrix $A_{3 \times 6}$ given by

$$A := \begin{bmatrix} 1 & 2 & -1 & 2 & -1 \\ 2 & 4 & 0 & 6 & 2 \\ 1 & 2 & 2 & 5 & 5 \end{bmatrix}.$$

The reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

w7.1a) Find a basis for Nul A. What is the dimension of this subspace of $\mathbb{R}^5$?

w7.1.b) Find a basis for Col A. Pick your basis so that it uses some (but not all!) of the columns of A. Hint: throw away columns that are dependent on the ones you don't throw away.

w7.1c) Verify that the dimensions of the two subspaces in parts a,b add up to 5, the number of columns of A. This is an example of a general fact, known as the "rank plus nullity theorem". To see what the rank plus nullity theorem is and why it's true, consider any matrix $B_{m \times n}$ which has m rows and n columns. Consider the associated linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\underline{\mathbf{x}}) = B\,\underline{\mathbf{x}}$. Then

$$Nul\,B = kernel\,T \subseteq \mathbb{R}^n\,.$$
$$Col\,B = range\,T \subseteq \mathbb{R}^m.$$

Let the reduced row echelon form of B have k pivots. Explain what the dimensions of $Nul\,B$ and $Col\,B$ are in terms of k and the total number of columns n, and then verify that

$$dim\,(Nul\,B) + dim\,(Col\,B) = n$$

must hold. Hint: these numbers are related to which columns of the reduced matrix have pivots, and which ones don't.

Remark: The dimension of $Col\,B$ above is called the column rank of the matrix B. The dimension of $Nul\,B$ is called the nullity. That nomenclature is why the theorem is called the "rank plus nullity theorem". You can read more about this theorem at wikipedia - a version holds for general linear transformations between vector spaces and has important applications.

w7.1d) If you have a collection of vectors $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n\}$ then it is relatively straightforward to check that the following three elementary operations do not change their span:

(i) interchanging two of the vectors.
(ii) replacing one of the vectors by a non-zero scalar multiple of itself.
(iii) replacing one of the vectors by the sum of it with a scalar multiple of a different one of the vectors.

Use these facts to compute the "reduced column echelon form" of the matrix A above, and deduce a "nicer" basis for $Col\,A$ than the one you found in part **b**. We will do an example like this in class. (It will be a nicer basis because there will be more zero entries in the basis vectors, and it will be easier to express any vector in $Col\,A$ as a linear combination of these improved basis vectors.)