Math 2270-004
Week 12-13 homework,
due November 21.

Recall, only the bold-faced underlined problems need to be handed in.

6.2 Orthogonal and orthonormal sets of vectors
7, **9**, 11, **13**, **17**.

6.3 Orthogonal projections
1, 3, **5**, **7**, **11**, **13**, **21**.

6.4 The Gram-Schmidt process
1, 3, **5**, **7**, **11**, 13, **15**.

w12.1a) Find the orthogonal complement $W^{\perp}$ for $W = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

b) Find $proj_W\, \underline{x}$ for $\underline{x} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$.

c) Express $\underline{x}$ as the sum of a vector in W with a vector in $W^{\perp}$.

w12.2a) Continue from **6.4.7**, to get an orthormal basis for $\mathbb{R}^3$ that is the result of Gram-Schmidt from the given set of three vectors:

$$\mathbb{R}^3 = span\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

b) Let Q be the orthogonal matrix which has your ortho-normal basis from part a as columns. Verify that the rows of Q are also ortho-normal.

c) Find the $A = Q\,R$ factorization arising from Gram-Schmidt, for the matrix

$$A = \begin{bmatrix} 2 & 4 & 1 \\ -5 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

w12.3) Find an $A = Q\,R$ decomposition for this matrix:

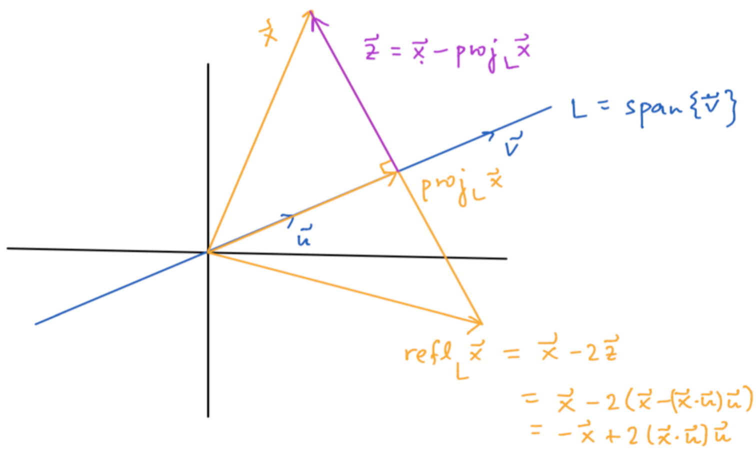
$$A = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}.$$

w12.4a) We know that in $\mathbb{R}^2$ projection onto $L = span\{\underline{\boldsymbol{v}}\}$ is given by the formula

$$proj_L \underline{\mathbf{x}} = (\underline{\mathbf{x}} \cdot \underline{\boldsymbol{u}})\underline{\boldsymbol{u}}$$

where $\underline{\boldsymbol{u}} = \dfrac{\underline{\boldsymbol{v}}}{\|\underline{\boldsymbol{v}}\|}$ is the unit vector in the direction of $\underline{\boldsymbol{v}}$. It follows that reflection across the line L is given by

$$refl_L \underline{\mathbf{x}} = \underline{\mathbf{x}} - 2\,\underline{\mathbf{z}} = -\underline{\mathbf{x}} + 2(\underline{\mathbf{x}} \cdot \underline{\boldsymbol{u}})\underline{\boldsymbol{u}}\,.$$



Write $\underline{\boldsymbol{u}} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$. Use the formula above to show that the linear transformation for reflection can also be expressed in matrix form as

$$refl_L \underline{\mathbf{x}} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Hint: Recall that the matrix for the transformation is given by $A = [\,T(\underline{\boldsymbol{e}}_1)\;\; T(\underline{\boldsymbol{e}}_2)\,]$. Compute the matrix columns $refl_L \underline{\boldsymbol{e}}_1$ and $refl_L \underline{\boldsymbol{e}}_2$ with the formula for $refl_L \underline{\mathbf{x}}$ and then use the double angle formulas.

b) Verify that if L is $span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, i.e. the line $x_2 = x_1$, then part a yields the correct matrix for reflection across this line which yields the formula $refl_L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$ which you probably recall from Calculus, when you discussed how graphs of functions and their inverse functions are related by reflecting across this line, and which you could figure out using earlier material in our class.

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