In HW, you're thinking abt: (1) what makes linear sys consistent / inconsistent (2) when do you get 1 or ∞'ly sol's for consistent systems

Exercise 4 We are interested in the matrix equation $A\,\mathbf{x} = \mathbf{b}$ for the matrix A below, and three different right hand sides at once.

$$\begin{array}{c} \text{vars} \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \\ A := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \end{array} \qquad rref(A) = \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let's consider three different linear systems for which A is the coefficient matrix. In the first one, the right hand sides are all zero (what we call the "homogeneous" problem), and I have carefully picked the other two right hand sides. The three right hand sides are separated by the dividing line below:

$$C := \left[\begin{array}{ccccc|ccc} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{array}\right] \qquad rref(C) = \left[\begin{array}{ccccc|c|c|c} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

4a) Find the solution sets for each of the three systems, using the reduced row echelon form of C.

sys. 1 $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$

sys 1

$x_1 + 2x_3 + x_4 + 3x_5 = 0$

$x_2 - 2x_3 - 3x_4 + x_5 = 0$

syst. 1

$x_1 = -2t_3 - t_4 - 3t_5$

$x_2 = 2t_3 + 3t_4 - t_5$

$x_3 = t_3 \in \mathbb{R}$

$x_4 = t_4 \in \mathbb{R}$

$x_5 = t_5 \in \mathbb{R}$ (free)

syst. 2

inconsistent (no solutions) 3rd eqtn requires $0x_1 + 0x_2 + \dots + 0x_5 = 1$

$x_1 = -2t_3 - t_4 - 3t_5$

$x_2 = 1 + 2t_3 + 3t_4 - t_5$

$x_3 \in t_3 \in \mathbb{R}$

$x_4 = t_4 \in \mathbb{R}$

$x_5 = t_5 \in \mathbb{R}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t_3 \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

same

these solutions are almost the same.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \bigcirc$$

$$C := \left[\begin{array}{ccccc|c|cc} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{array}\right] \qquad rref(C) = \left[\begin{array}{ccccc|c|cc} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Important conceptual questions:

4b) Which of these three solutions on the previous page could you have written down just from the reduced row echelon form of A, i.e. without using the augmented matrix and the reduced row echelon form of the augmented matrix? Why?

1st one, i.e. the homogeneous problem.
because if $\vec{b} = \vec{0}$ then that (augmented) column stays $\vec{0}$ when we do el. row ops

$A\vec{x} = \vec{0}$

4c) Linear systems in which right hand side vectors equal zero are called homogeneous linear systems. Otherwise they are called inhomogeneous or nonhomogeneous. Notice that the general solution to the consistent inhomogeneous system is the sum of a particular solution to it, together with the general solution to the homogeneous system!!! This is a theorem. Can you see why it's true?

later.

4d) In general, can you tell how many free parameters the solutions to a matrix system $A\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$ will have, based on the reduced row echelon form of A alone (assuming the system is consistent, i.e. has at least one solution)? State what's true and explain why!

$\underbrace{\#\text{ free params}}_{\text{i.e. free variables}} = \#$ of non-pivot col's in rref(A)

Wed Aug 29

- 1.5 solution sets to matrix equations; homogeneous and nonhomogeneous systems of equations, continued.

<u>Announcements:</u>

- part of next week's hw:
  - 1.4 5, 7, 11, 13, 17, 19, 21, 22, 23, 24, 25, 26, 31
  - 1.5 5, 11, 13, 17, 21, 23

  more to come ...
- quiz day ☺

'til 12:58

<u>Warm-up Exercise</u>:

a) Do the vectors in the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ span $\mathbb{R}^3$?

i.e. is $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^3$?

you may use the fact that

$$\begin{bmatrix} 1 & 4 & 1 \\ 2 & -1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

i.e. can I solve

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

for each $\vec{b} \in \mathbb{R}^3$?

YES

synthetic

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & b_1 \\ 2 & -1 & 0 & b_2 \\ 3 & 0 & 0 & b_3 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right]$$

same el. row ops

c's made out of b_1, b_2, b_3

take $x_1 = c_1$
$x_2 = c_2$
$x_3 = c_3$

b) Do the vectors in the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -6 \end{bmatrix} \right\}$ span $\mathbb{R}^3$?

NO

you may use the fact that

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & -1 & -5 \\ 3 & 0 & -6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -5 \\ -6 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & -1 & -5 & b_2 \\ 3 & 0 & -6 & b_3 \end{array}\right] \xrightarrow[\text{same el row ops}]{} \left[\begin{array}{ccc|c} 1 & 0 & -2 & c_1 \\ 0 & 1 & 1 & c_2 \\ 0 & 0 & 0 & c_3 \end{array}\right]$$

unless $c_3 = 0$ this system is inconsistent.
(so $\vec{b}$ is not in the span of the set)

In fact, the three vectors only spanned a plane in $\mathbb{R}^3$

so, pick any $\vec{c}$ with $c_3 \neq 0$.
Do reverse ("inverse") elem row ops

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & B_1 \\ 2 & -1 & -5 & B_2 \\ 3 & 0 & -6 & B_3 \end{array}\right] \leftarrow \leftarrow \leftarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & c_1 \\ 0 & 1 & 1 & c_2 \\ 0 & 0 & 0 & c_3 \end{array}\right]$$

these $\vec{B}$'s are not in the span.