What we may have realized in the previous exercise is the very important:

<u>Fundamental Fact</u> A *vector equation* (linear combination problem)

$$x_1\,\underline{\boldsymbol{a}}_1 + x_2\,\underline{\boldsymbol{a}}_2 + \ldots + x_n\,\underline{\boldsymbol{a}}_n = \underline{\boldsymbol{b}} \qquad \text{in } \mathbb{R}^m$$

is equivalent to a system of linear equations for the unknown weights $x_1, x_2, \ldots.\, x_n$; in fact the system of linear equations has augmented matrix given by

$$\left[\underline{\boldsymbol{a}}_1 \;\; \underline{\boldsymbol{a}}_2 \;\cdots\; \underline{\boldsymbol{a}}_n \;\; \underline{\boldsymbol{b}}\right]$$

(where we have expressed the augmented matrix in terms of its columns). In particular, $\underline{\boldsymbol{b}}$ can be generated by a linear combination of $\underline{\boldsymbol{a}}_1 \;\; \underline{\boldsymbol{a}}_2 \;\ldots\; \underline{\boldsymbol{a}}_n$ if and only if there exists a solution to the linear system corresponding to the augmented matrix above. Once we recognize the equivalence we can answer any question about a linear combination vector equation using Gaussian elimination and reduced row echelon form computations and concepts.

This fundamental fact is so important to the course, that we should check it in general at some point.

$$= x_1\begin{bmatrix} a_{11}\\ a_{21}\\ a_{31}\\ \vdots\\ a_{m1}\end{bmatrix} + x_2\begin{bmatrix} a_{12}\\ a_{22}\\ \vdots\\ a_{m2}\end{bmatrix} + \ldots + x_n\begin{bmatrix} a_{1n}\\ a_{2n}\\ \vdots\\ a_{mn}\end{bmatrix} = \begin{bmatrix} x_1 a_{11}\\ x_1 a_{21}\\ \vdots\\ x_1 a_{m1}\end{bmatrix} + \begin{bmatrix} x_2 a_{12}\\ x_2 a_{22}\\ \vdots\\ x_2 a_{m2}\end{bmatrix} + \ldots + \begin{bmatrix} x_n a_{1n}\\ x_n a_{2n}\\ \vdots\\ x_n a_{mn}\end{bmatrix}$$

§1.1-1.2 linear system

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n\\ \vdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n\end{bmatrix} \overset{\text{want}}{=} \begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_m\end{bmatrix}$$

<u>Exercise 3a</u>) Does the vector equation

$$x_1\begin{bmatrix}1\\0\\2\end{bmatrix} + \cancel{x_1}\, x_2\begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}2\\-3\\1\end{bmatrix}$$

have any solutions?

we did this in warmup $x_1 = 1/2$, $x_2 = -3/2$

<u>3b</u>) What geometric question is this related to? What geometric object is $span\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\0\end{bmatrix}\right\}$?

expect plane

3c) Use an augmented matrix calculation to find what condition needs to hold on vectors $\underline{\boldsymbol{b}}$ so that $\underline{\boldsymbol{b}} \in span\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\0\end{bmatrix}\right\}$. (!!) How does this computation relate to the (implicit) way we've been expressing planes in $\mathbb{R}^3$?

for which $\vec{b}$??

to see when we can solve

$$x_1\begin{bmatrix}1\\0\\2\end{bmatrix} + x_2\begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & -1 & b_1\\ 0 & 2 & b_2\\ 2 & 0 & b_3\end{array}\right]$$

$$\begin{array}{r}R_3/2 \to R_1\\ R_2/2 \to R_2\\ R_1\end{array}\left[\begin{array}{cc|c}1 & 0 & b_3/2\\ 0 & 1 & b_2/2\\ 1 & -1 & b_1\end{array}\right]$$

$$\begin{array}{r}\\ \\ -R_1+R_3 \to R_3\end{array}\left[\begin{array}{cc|c}1 & 0 & b_3/2\\ 0 & 1 & b_2/2\\ 0 & -1 & b_1 - b_3/2\end{array}\right]$$

$$\begin{array}{r}\\ \\ R_2+R_3 \to R_3\end{array}\left[\begin{array}{cc|c}1 & 0 & b_3/2\\ 0 & 1 & b_2/2\\ 0 & 0 & b_1 - b_3/2 + b_2/2\end{array}\right]$$

$$\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

consistent

$$b_1 - \frac{b_3}{2} + \frac{b_2}{2} = 0$$

$$x - \frac{z}{2} + \frac{y}{2} = 0$$

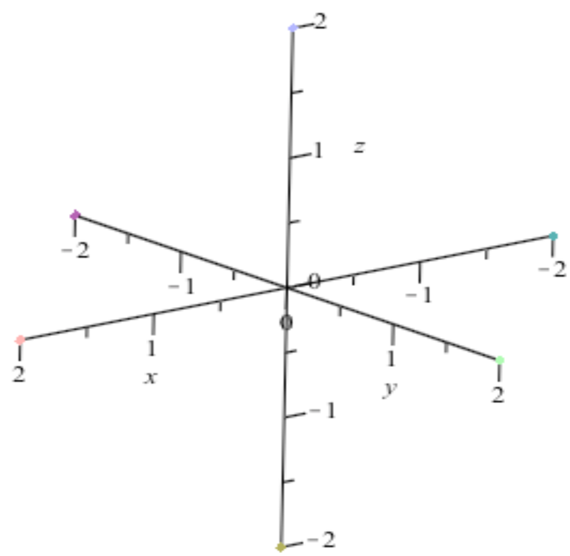
$$\boxed{2x + y - z = 0}$$

also $x_1 = b_3/2$, $x_2 = b_2/2$

$$\begin{bmatrix}2\\-3\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\0\\2\end{bmatrix} - \frac{3}{2}\begin{bmatrix}-1\\2\\0\end{bmatrix}$$

inconsistent if $2x + y - z \neq 0$

In case we want to sketch anything related to Exercise 3:



Tues Aug 28

- 1.4 the matrix equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$. How the reduced row echelon form of (just) A relates to solvability questions (leads into section 1.5).

<u>Announcements:</u>

- MTW 2-2:50 LCB 225 office hours
  (I need to change on syllabus)
- HW due tomorrow
- Quiz as well 1.1-1.3

'til 12:57

<u>Warm-up Exercise</u>: Is $\begin{bmatrix}2\\-3\\1\end{bmatrix}$ in span$\left\{ \underset{\vec{u}}{\begin{bmatrix}1\\0\\2\end{bmatrix}}, \underset{\vec{v}}{\begin{bmatrix}-1\\2\\0\end{bmatrix}} \right\}$? (Exercise 3a yesterday's notes)

Hint: the vector equation you're trying to solve is

$$* \qquad x_1\begin{bmatrix}1\\0\\2\end{bmatrix} + x_2\begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}2\\-3\\1\end{bmatrix}$$

yes: $\frac{1}{2}\begin{bmatrix}1\\0\\2\end{bmatrix} - \frac{3}{2}\begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}2\\-3\\1\end{bmatrix}$ ✓

How? * vector eqtn is equiv. to linear sys. with augmented matrix

$$\left[\begin{array}{cc|c}1 & -1 & 2\\ 0 & 2 & -3\\ 2 & 0 & 1\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 0 & 1/2\\ 0 & 1 & -3/2\\ 0 & 0 & 0\end{array}\right]$$

terminology:

<u>span</u> $\{\vec{u},\vec{v}\}$:= collection of all <u>linear combinations of $\vec{u},\vec{v}$</u>

any $\vec{w} = c_1\vec{u} + c_2\vec{v}$ $\quad c_1, c_2 \in \mathbb{R}$

(or $x_1\vec{u} + x_2\vec{v}$)

Recall ✓

<u>Fundamental Fact</u> A *vector equation* (linear combination problem)

$$x_1 \underline{\boldsymbol{a}}_1 + x_2 \underline{\boldsymbol{a}}_2 + \ldots + x_n \underline{\boldsymbol{a}}_n = \underline{\boldsymbol{b}}$$

is actually a system of linear equations for the unknown weights $x_1, x_2, \ldots x_n$; in fact the system of linear equations has augmented matrix given by

$$\left[\underline{\boldsymbol{a}}_1 \ \ \underline{\boldsymbol{a}}_2 \ \cdots \ \underline{\boldsymbol{a}}_n \ \ \underline{\boldsymbol{b}}\ \right]$$

(where we have expressed the augmented matrix in terms of its columns). In particular, $\underline{\boldsymbol{b}}$ can be generated by a linear combination of $\underline{\boldsymbol{a}}_1 \ \ \underline{\boldsymbol{a}}_2 \ \ldots \ \underline{\boldsymbol{a}}_n$ if and only if there exists a solution to the linear system corresponding to the augmented matrix above.

We should check this carefully today, assuming we didn't do so on Monday:

<u>Definition</u> (from 1.4) If A is an $m \times n$ matrix, with columns $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots \underline{\boldsymbol{a}}_n$ (in $\mathbb{R}^m$) and if $\underline{\boldsymbol{x}} \in \mathbb{R}^n$, then $A\,\underline{\boldsymbol{x}}$ is defined to be the linear combination of the columns, with weights given by the corresponding entries of $\underline{\boldsymbol{x}}$. In other words,

$$A\,\underline{\boldsymbol{x}} := x_1 \underline{\boldsymbol{a}}_1 + x_2 \underline{\boldsymbol{a}}_2 + \ \ldots x_n \underline{\boldsymbol{a}}_n \,.$$

(This will give us a way to abbreviate vector equations, for example.)

$$\begin{bmatrix} 1 & 3 & -6 \\ 2 & 4 & 17 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} -6 \\ 17 \end{bmatrix}$$

Definition. Let $\underline{u}, \underline{v}$ be vectors in $\mathbb{R}^n$. Then the *dot product* $\underline{u} \cdot \underline{v}$ is defined by

$$\underline{u} \cdot \underline{v} = \sum_{j=1}^{n} u_j v_j = u_1 v_1 + u_2 v_2 + \ldots u_n v_n .$$

$$\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} = 1\cdot 3 + 4\cdot(-1) + (-2)\cdot 6$$
$$= 3 - 4 - 12 = -13$$

Computational Theorem: (This is usually a quicker way to compute $A\,\underline{x}$. Let If A be an $m \times n$ matrix, with rows $R_1, R_2, \ldots R_m$. Then $A\,\underline{x}$ may also be computed using the rows of A and the dot product:

def OR

$$x_1 \underline{a}_1 + x_2 \underline{a}_2 + \ldots x_n \underline{a}_n = A\,\underline{x} = \begin{bmatrix} R_1 \cdot \underline{x} \\ R_2 \cdot \underline{x} \\ \vdots \\ R_m \cdot \underline{x} \end{bmatrix}$$

go look at the fact: vector eqtns = linear systems

Exercise 1a) Compute both ways:

$$\begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \end{bmatrix}$$

OR

$$= \begin{bmatrix} 1\cdot 2 + (-2)(-2) + 3\cdot 1 \\ (-2)(2) + 3(-2) + 4\cdot 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \end{bmatrix}$$

Exercise 1b) Write as a matrix times a vector:

$$3\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -1 \\ 1 & 3 & 2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

Exercise 2) Rewrite the following vector equations from yesterday and last week as matrix equations. Also write down the augmented matrix for these systems.

2a)

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ -1 & 3 & 8 \end{array}\right]$$

2b)

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_1 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

What the solvability and number of solutions to a matrix equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ has to do with the reduced row echelon form of A (i.e. of the unaugmented matrix). Let's explore.

<u>Exercise 3</u> Find all solutions to the system of 3 linear equations in 5 unknowns

(skip in class... this is review of algorithm for rref.)

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27\,. \end{aligned}$$

Here's the augmented matrix:

$$\left[\begin{array}{rrrrr|r} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right]$$

Find the reduced row echelon form of this augmented matrix and then backsolve to explicitly parameterize the solution set. (Hint: it's a two-dimensional plane in $\mathbb{R}^5$, if that helps. :-))

Maple says:

```
> with(LinearAlgebra) :   # matrix and linear algebra library
> A := Matrix(3, 5, [1, -2, 3, 2, 1,
                     2, -4, 8, 3, 10,
                     3, -6, 10, 6, 5]) :
  b := Vector([10, 7, 27]) :
  <A|b>;     # the mathematical augmented matrix doesn't actually have
               # a vertical line between the end of A and the start of b
  ReducedRowEchelonForm(<A|b>);
```

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix} \qquad \textbf{(1)}$$

```
> LinearSolve(A, b);
       #  this command will actually write down the general solution, using
       # Maple's way of writing free parameters, which actually makes
       # some sense.  Generally when there  are free parameters involved,
       #  there will be equivalent ways to express the solution that may
       # look different.  But usually Maple's version will look like yours,
       # because it's using the same algorithm and choosing the free parameters
       # the same way too.
```

$$\begin{bmatrix} 5 + 2\,\_t_2 - 3\,\_t_5 \\ \_t_2 \\ -3 - 2\,\_t_5 \\ 7 + 4\,\_t_5 \\ \_t_5 \end{bmatrix} \qquad \textbf{(2)}$$

```
>
```

In HW, you're thinking abt: ① what makes linear sys consistent / inconsistent ② when do you get 1 or ∞'ly sol's for consistent systems

Exercise 4 We are interested in the matrix equation $A\,\mathbf{x} = \boldsymbol{b}$ for the matrix A below, and three different right hand sides at once.

$$\begin{array}{c} \text{vars} \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \end{array}$$

$$A := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \qquad rref(A) = \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let's consider three different linear systems for which A is the coefficient matrix. In the first one, the right hand sides are all zero (what we call the "homogeneous" problem), and I have carefully picked the other two right hand sides. The three right hand sides are separated by the dividing line below:

$$C := \left[\begin{array}{ccccc|ccc} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{array}\right] \qquad rref(C) = \left[\begin{array}{ccccc|c|c|c} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

4a) Find the solution sets for each of the three systems, using the reduced row echelon form of C.

sys. 1 $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$

sys 1

$x_1 + 2x_3 + x_4 + 3x_5 = 0$

$x_2 - 2x_3 - 3x_4 + x_5 = 0$

syst. 1

$x_1 = -2t_3 - t_4 - 3t_5$

$x_2 = 2t_3 + 3t_4 - t_5$

$x_3 = t_3 \in \mathbb{R}$

$x_4 = t_4 \in \mathbb{R}$

$x_5 = t_5 \in \mathbb{R}$ (free)

syst. 2

inconsistent (no solt'ns)

3rd eq'n requires

$0x_1 + 0x_2 + \dots + 0x_5 = 1$

$x_1 = -2t_3 - t_4 - 3t_5$

$x_2 = 1 + 2t_3 + 3t_4 - t_5$

$x_3 = t_3 \in \mathbb{R}$

$x_4 = t_4 \in \mathbb{R}$

$x_5 = t_5 \in \mathbb{R}$

these solutions are almost the same.