Math 2270-002 Week 2 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we plan to cover. These notes cover material in 1.3-1.6. They include material from last weeks notes that we did not get to.
1.3 vector equations
1.4 matrix equations encompass vector equations and linear systems of equations
1.5 structure of solution sets to matrix equations
1.6 some applications

Mon Aug 27
- 1.3 algebra and geometry for vector equations and linear combinations

Announcements:
- returned quizzes are in folders (solutions on CANVAS)
- new notes for week 2
- M, T office hours request here LCB 215 → hour after class (T class LCB 225 not available)

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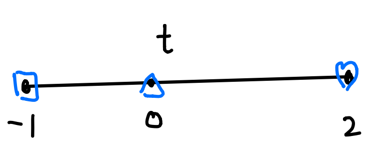
Warm-up Exercise: Sketch the line segment of points, whose position vectors are given by

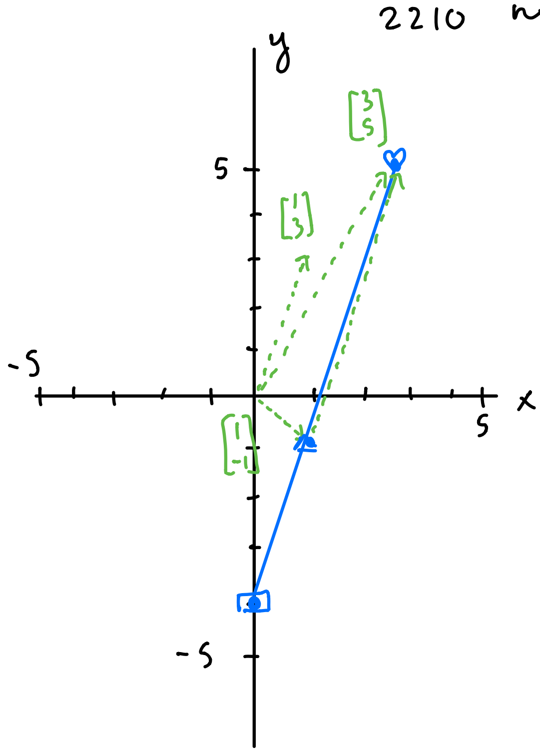
$$\vec{r}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix} = (1+t)\hat{i} + (-1+3t)\hat{j} = \underbrace{\langle 1+t, -1+3t \rangle}_{2210 \text{ notation}}$$

for $-1 \le t \le 2$

or $x = 1+t$
$y = -1+3t$
$-1 \le t \le 2$

t	$\vec{r}(t)$	(x,y)	
0	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$(1,-1)$	
-1	$\begin{bmatrix} 0 \\ -4 \end{bmatrix}$	$(0,-4)$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 3 \end{bmatrix}$
2	$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$	$(3,5)$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \end{bmatrix}$





On Friday we defined vectors algebraically, as ordered lists of numbers. And, we defined vector addition and scalar multiplication, which you've worked with in previous courses, although maybe only in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$:

<u>Definition:</u> For $\underline{\boldsymbol{u}} = \begin{bmatrix} u_1 \\ u_2 \\ : \\ u_n \end{bmatrix}, \underline{\boldsymbol{v}} = \begin{bmatrix} v_1 \\ v_2 \\ : \\ v_n \end{bmatrix} \in \mathbb{R}^n; c \in \mathbb{R}$, then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ : \\ u_n + v_n \end{bmatrix}$; $c\,\underline{\boldsymbol{u}} := \begin{bmatrix} cu_1 \\ cu_2 \\ : \\ cu_n \end{bmatrix}$.

There are a number of straightforward algebra identities for vector addition and scalar multiplication. These all reduce to real number axioms when one looks at the individual entries of the vectors on each side of the identities:

Let $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in \mathbb{R}^n$, $c, d \in \mathbb{R}$. Then

(i) $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{u}}$

e.g. $\text{entry}_i(\vec{u}+\vec{v}) = u_i + v_i$
$\text{entry}_i(\vec{v}+\vec{u}) = v_i + u_i$
> equal because real # addition is commutative

(ii) $(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) + \underline{\boldsymbol{w}} = \underline{\boldsymbol{u}} + (\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}})$

$\text{entry}_i((\vec{u}+\vec{v})+\vec{w}) = (u_i+v_i)+w_i$
$\text{entry}_i(\vec{u}+(\vec{v}+\vec{w})) = u_i + (v_i+w_i)$
> equal because real # addition is associative

(iii) $\underline{\boldsymbol{u}} + \underline{\boldsymbol{0}} = \underline{\boldsymbol{0}} + \underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$ ($\underline{\boldsymbol{0}}$ is defined to be the vector for which each entry is the number 0.)

(iv) $\underline{\boldsymbol{u}} + (-\underline{\boldsymbol{u}}) = (-\underline{\boldsymbol{u}}) + \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}}$ ($-\underline{\boldsymbol{u}}$ is defined to be $-1 \cdot \underline{\boldsymbol{u}}$, i.e. the vector for which each entry is the opposite of the corresponding entry in $\underline{\boldsymbol{u}}$.)

(v) $c(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = c\,\underline{\boldsymbol{u}} + c\,\underline{\boldsymbol{v}}$

(vi) $(c + d)\underline{\boldsymbol{u}} = c\,\underline{\boldsymbol{u}} + d\,\underline{\boldsymbol{u}}$

(vii) $c(d\,\underline{\boldsymbol{u}}) = (c\,d)\,\underline{\boldsymbol{u}}$

(viii) $1\,\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}$.

Geometric interpretation of vectors as displacements

The space $\mathbb{R}^n$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^n$ consists of all possible $n - tuples$ of numbers:

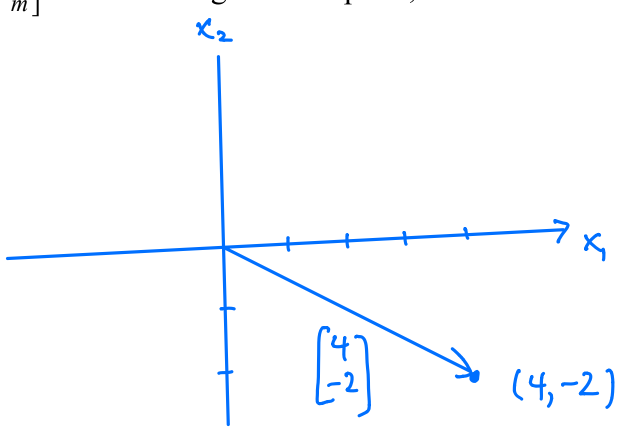
(i) We can think of those $n - tuples$ as representing points, as we're used to doing for $n = 1, 2, 3$. In this case we can write

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n), s.t.\ x_1, x_2, ..., x_n \in \mathbb{R}\}.$$

(ii) We can think of those $n - tuples$ as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ s.t.\ x_1, x_2, ..., x_n \in \mathbb{R} \right\}.$$

Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^n$ as sets by identifying each point $(x_1, x_2, ... x_n)$ in the first model with the displacement vector $\underline{\mathbf{x}} = [x_1, x_2, ... x_m]^T$ from the origin to that point, in the second model, i.e. the "position vector" of the point.



Exercise 1 : Finish this exercise from Friday ...

Exercise 3) (Friday) Let $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

3a) Plot the points $(1, -1)$ and $(1, 3)$, which have position vectors $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$. Draw these position vectors as arrows beginning at the origin and ending at the corresponding points.

3b) Compute $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$ and then plot the point for which this is the position vector. Note that the algebraic operation of vector addition corresponds to the geometric process of composing horizontal and vertical displacements.

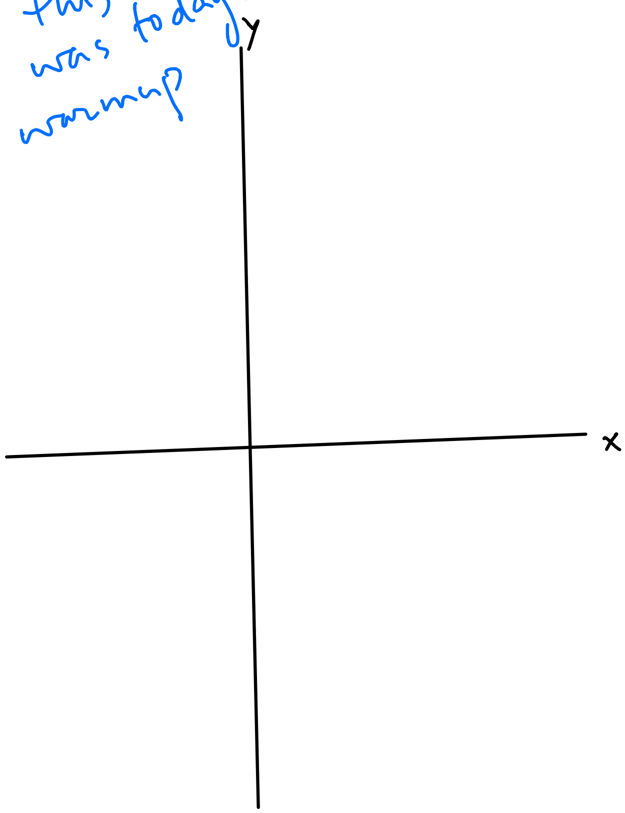
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

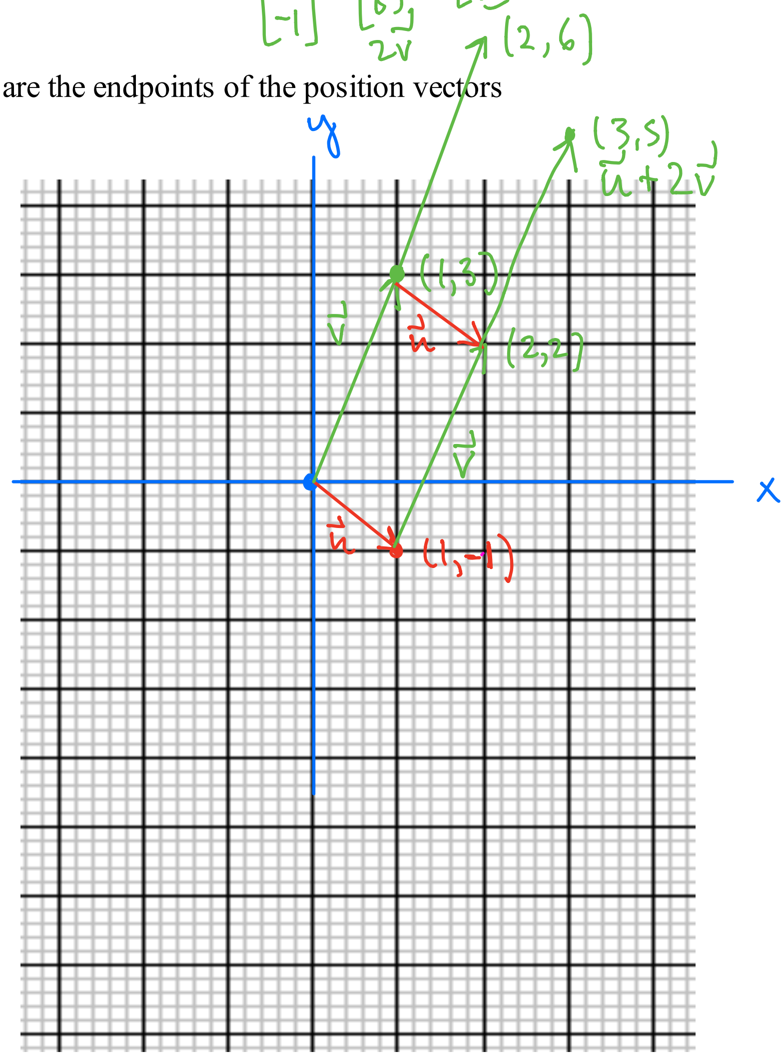
3c) Compute $\underline{\boldsymbol{u}}$ and $2\,\underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{u}} + 2\,\underline{\boldsymbol{v}}$ plot the corresponding points for which these are the position vectors.

$$2\vec{v} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \qquad \vec{u} + 2\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 6 \end{bmatrix}}_{2\vec{v}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

3d) Plot the parametric line segment whose points are the endpoints of the position vectors $\{\underline{\boldsymbol{u}} + t\,\underline{\boldsymbol{v}},\ -1 \le t \le 2\}$.

this was today's warmup





One of the key themes of this course is the idea of "linear combinations". These have an algebraic definition, as well as a geometric interpretation as combinations of displacements.

Definition: If we have a collection of n vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p\}$ in $\mathbb{R}^n$, then any vector $\underline{v} \in \mathbb{R}^n$ that can be expressed as a sum of scalar multiples of these vectors is called a *linear combination* of them. In other words, if we can write

$$\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_p\underline{v}_p \,,$$

then $\underline{v}$ is a *linear combination* of $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the *linear combination coefficients* or *weights*.

Example You've probably seen linear combinations in previous math/physics classes, even if you didn't realize it. For example you might have expressed the position vector $\underline{\mathbf{r}}$ as a linear combination

$$\underline{\mathbf{r}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}} \quad = \langle x, y, z \rangle$$

where $\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$ represent the unit displacements in the x, y, z directions. Since we can express these displacements using Math 2270 notation as

$$\underline{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we have

$$x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Exercise 2) Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the $(\pm)$ directions of $\underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

2a) Superimpose a grid related to the displacement vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.

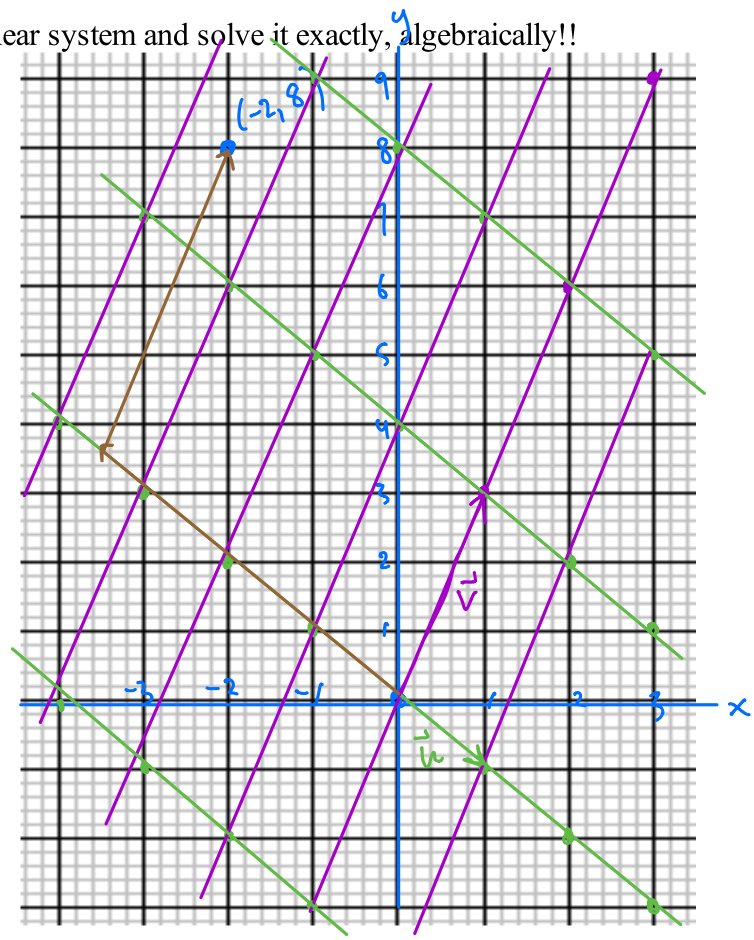
2b) Rewrite the linear combination problem as a linear system and solve it exactly, algebraically!!

geometric guess

$$\begin{bmatrix} -2 \\ 8 \end{bmatrix} \approx -3.5\vec{u} + 1.5\vec{v}$$

$$\overset{?}{=} -3.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 8 \end{bmatrix} \; !$$



$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 &= -2 \\ -x_1 + 3x_2 &= 8 \end{aligned}$$

Lin Sys!

$$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ -1 & 3 & 8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 4 & 6 \end{array}\right] \quad R_1 + R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 1 & 1.5 \end{array}\right] \quad R_2/4 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & -3.5 \\ 0 & 1 & 1.5 \end{array}\right] \quad -R_2 + R_1 \to R_1$$

algebraically, $x_1 = -3.5$

$x_2 = 1.5$

agrees.

2c) Can you get to any point (x, y) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$?

Argue geometrically and algebraically. How many ways are there to express $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$?

geometric reason: $\vec{u}, \vec{v}$ are not parallel, so the "$\vec{u}, \vec{v}$" grid covers all of $\mathbb{R}^2$.

algebraic reason:

instead of (-2, 8)

vector eqtn is $x_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

augmented matrix $\left[\begin{array}{cc|c} 1 & 1 & x \\ -1 & 3 & y \end{array}\right]$ reduces to $\left[\begin{array}{cc|c} 1 & 0 & c_1 \\ 0 & 1 & c_2 \end{array}\right]$

so $x_1 = c_1$, $x_2 = c_2$

made out of x's & y's.

Definition The *span* of a collection of vectors, written as $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$, is the collection of all linear combinations of those vectors.

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Examples using this language:

- We showed in 2c that $span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\} = \mathbb{R}^2$

$span\{\vec{u}, \vec{v}\} := \{\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 \text{ s.t. } c_1, c_2 \in \mathbb{R}\}$

such that

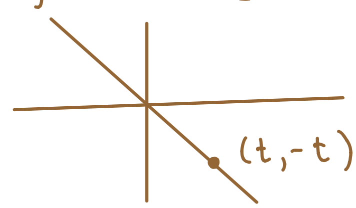
"is defined to be"

- The other hand, $span\{\underline{\boldsymbol{u}}\}$ is the line with implicit equation $y = -x$.

$span\{\vec{u}\} := \{c_1\vec{u} \text{ s.t. } c_1 \in \mathbb{R}\} = \{t\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ s.t. } t \in \mathbb{R}\}$

unlike "wingspan", vector "span" continues indefinitely

- in $\mathbb{R}^3$, $span\{\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}}, \underline{\boldsymbol{k}}\} = \mathbb{R}^3$.



pt. with pos. vector $\begin{bmatrix} t \\ -t \end{bmatrix}$

Remark: The mathematical meaning of the word *span* is related to the English meaning - as in "wing span" or "span of a bridge", but it's also different. The span of a collection of vectors goes on and on and does not "stop" at the vector or associated endpoint.