2 linear equations in 2 unknowns:

$$a_{11}\,x + a_{12}\,y = b_1$$
$$a_{21}\,x + a_{22}\,y = b_2$$

goal: find all $[x, y]$ making both of these equations true at once. Since the solution set to each single equation is a line one geometric interpretation is that you are looking for the intersection of two lines.

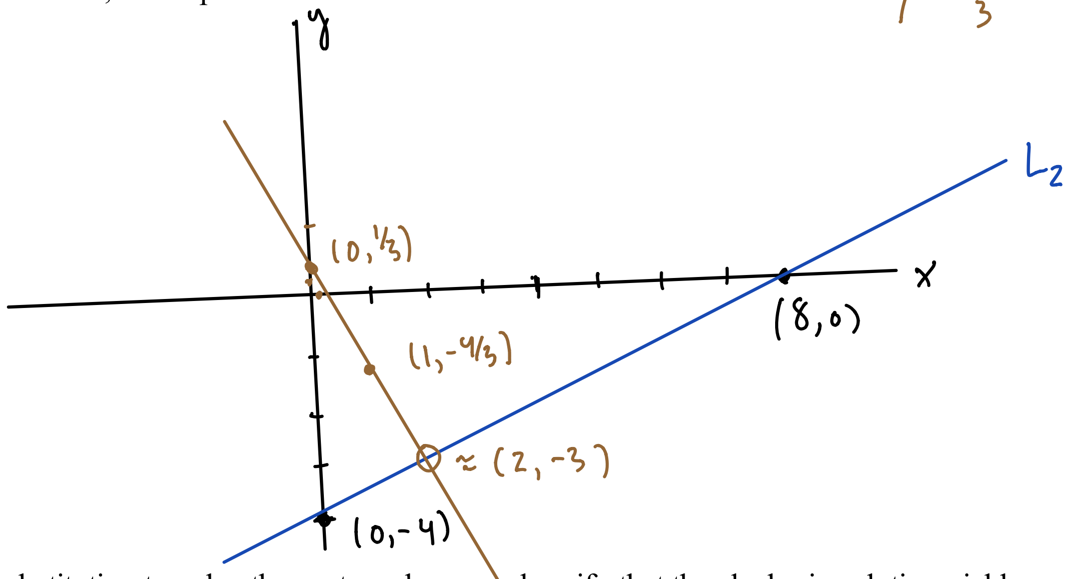
Exercise 3: Consider the system of two equations E_1, E_2:

$$E_1 \qquad 5\,x + 3\,y = 1$$
$$E_2 \qquad x - 2\,y = 8$$

$3y = 1 - 5x \quad y = \frac{1}{3} - \frac{5}{3}x$

$x = 1$
$y = -\frac{4}{3}$

3a) Sketch the solution set in $\mathbb{R}^2$, as the point of intersection between two lines.



3b) Use the method of substitution to solve the system above, and verify that the algebraic solution yields the point you estimated geometrically in part a.

$E_2 \quad x = 8 + 2y$

$E_1 \to \quad 5(8 + 2y) + 3y = 1$

$13y + 40 = 1$

$13y = -39, \Rightarrow y = -3; \quad x = 8 - 6 = 2.$

$x = 2$
$y = -3$

works great for all sorts of systems, as long as only 2 variables

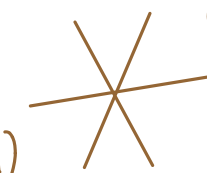
with more than 2 variables substitution gets messy fast

3c) What are the possible geometric solution sets to 1, 2, 3, 4 or any number of linear equations in two unknowns?

- ∞'ly many solutions: each eqtn had same line as its solution set

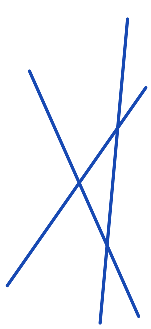
e.g $\begin{cases} 5x + 3y = 1 \\ 10x + 6y = 2 \end{cases}$

- 1 solution: there is one point (& no more) on all of the lines



- no solutions e.g. 2 parallel lines (or at least 2 of the lines are parallel)

OR!



Math 2270-002
Tues Aug 21
- 1.1 Systems of linear equations and Gaussian elimination
- 1.2 Row reduction, echelon form, and reduced row echelon form

Announcements:
- Our Canvas page is live.
- I've posted HW #1 on public page & on CANVAS ← Start HW soon!
- Chapter 1 of text is on CANVAS
- Quiz 1 tomorrow ☺ on Gaussian elimination

'til 12:58

Warm-up Exercise: a) Sketch the two lines

$$\begin{cases} 5x + 3y = 1 \\ x - 2y = 8 \end{cases}$$

and estimate their point of intersection from your sketch.

b) Solve the system in (a) algebraically. Compare to sketch

(this is Exercise 3 in yesterday's notes ~ so you can do your work there.)

3ab

b solution $(x, y) = (2, -3)$ writing as a point

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$ writing as position vectors

This is the system we solved at the end of class on Monday:

$$\begin{cases} E_1 & 5x + 3y = 1 \\ E_2 & x - 2y = 8 \end{cases}$$

The method of substituion that we used then does not work well for systems of equations with lots of variables and lots of equations. What does work well is *Gaussian elimination*, which we now explain and illustrate.

great for linear systems with more than two variables

1a) Use the following three "elementary equation operations" to systematically reduce the system $\{E_1, E_2\}$ to an equivalent system (i.e. one that has the same solution set), but of the form

$$1\,x + 0\,y = c_1$$
$$0\,x + 1\,y = c_2$$

(so that the solution is $x = c_1, y = c_2$). Make sketches of the intersecting lines, at each stage.

The three types of elementary equation operation that we use are listed below. Can you explain why the solution set to the modified system is the same as the solution set before you make the modification?

- interchange the order of the equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

} won't change soltn set

does not effect whether all of the eqtns are true.

if $a = a$ then $ca = ca$. If $ca = ca$ & $c \neq 0 \Rightarrow a = a$ (divide by c)

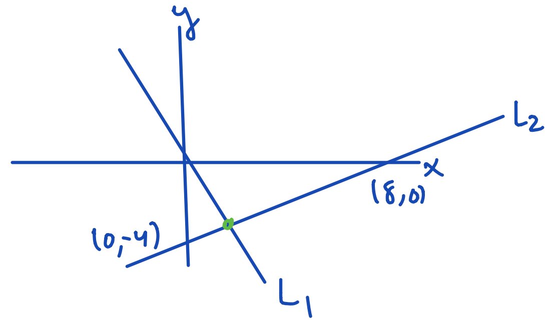
if $\begin{matrix} a = a \\ b = b \end{matrix}$ then $\begin{matrix} a = a \\ cb = cb \end{matrix}$ then $a + cb = a + cb$ is true.

reversible: if $\begin{cases} a + cb = a + cb \\ b = b \end{cases} \Rightarrow \begin{array}{r} a + cb = a + cb \\ - \quad cb = cb \\ \hline a = a. \end{array}$

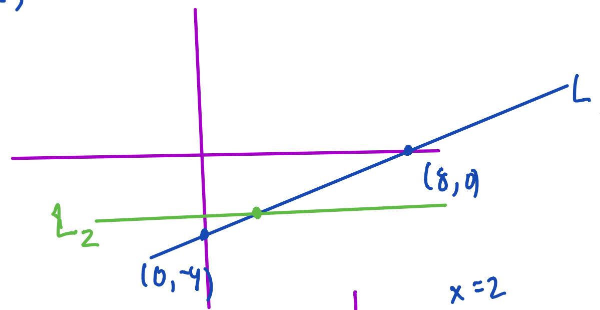
$$\begin{aligned}
E_1 &\quad 5x + 3y = 1 \\
E_2 &\quad x - 2y = 8
\end{aligned}$$

$$\begin{aligned}
E_2 \to E_1 &\quad 1 \cdot x - 2y = 8 \\
E_1 \to E_2 &\quad 5x + 3y = 1
\end{aligned}$$

changed names of lines



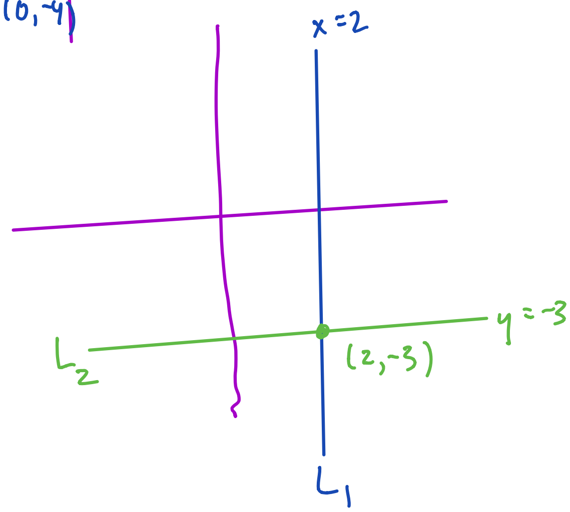
$$\begin{aligned}
&\quad x - 2y = 8 \\
-5E_1 + E_2 \to E_2 &\quad 13y = -39
\end{aligned}$$



$$\begin{aligned}
&\quad x - 2y = 8 \\
\tfrac{1}{13}E_2 \to E_2 &\quad 1 \cdot y = -3
\end{aligned}$$

$$\begin{aligned}
2E_2 + E_1 \to E_1 &\quad x = 2 \\
&\quad y = -3
\end{aligned}$$

$x = 2$
$y = -3$



1b) Look at our work in 1a. Notice that we could have saved a lot of writing by doing this computation "synthetically", i.e. by just keeping track of the coefficients in front of the variables and right-side values: We represent the system

$$\begin{array}{ll} E_1 & 5x + 3y = 1 \\ E_2 & x - 2y = 8 \end{array}$$

with the *augmented matrix*

coeff matrix

augment with right side values

$$\left[\begin{array}{cc|c} 5 & 3 & 1 \\ 1 & -2 & 8 \end{array}\right].$$

x-coefs y coef's

Using R_1, R_2 as symbols for the rows, redo the operations from part (a). Notice that when you operate synthetically the "elementary equation operations" correspond to "elementary row operations":

- interchange two rows
- multiply a row by a non-zero number
- replace a row by its sum with a multiple of another row.

$$\begin{array}{cc|c} 5 & 3 & 1 \\ 1 & -2 & 8 \end{array}$$

$R_2 \to R_1$, $R_1 \to R_2$

$$\begin{array}{cc|c} 1 & -2 & 8 \\ 5 & 3 & 1 \end{array}$$

$-5R_1 + R_2 \to R_2$

$$\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 13 & -39 \end{array}$$

$\frac{R_2}{13} \to R_2$

$$\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 1 & -3 \end{array}$$

$2R_2 + R_1 \to R_1$

$$\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}$$

uncloak

$1x + 0y = 2$
$0x + 1y = -3$

$x = 2$
$y = -3$

Solutions to linear equations in 3 unknowns:

What is the geometric question we're answering in these cases?

Exercise 1) Consider the system

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 9z &= 23 . \end{aligned}$$

Use elementary equation operations (or if you prefer, elementary row operations in the synthetic version) to find the solution set to this system. The systematic algorithm we use is called Gaussian elimination. (Check Wikipedia about Gauss - he was amazing!)

Hint: The solution set is a single point, $[x, y, z] = [5, -2, 3]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

Exercise 2) There are other possibilities. In the two systems below we kept all of the coeffients the same as in Exercise 1, except for a_{33} , and we changed the right side in the third equation, for 2a. Work out what happens in each case.

2a)

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 8z &= 20 . \end{aligned}$$

2b)

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 8z &= 23 . \end{aligned}$$

2c) What are the possible solution sets (and geometric configurations) for 1, 2, 3, 4,... equations in 3 unknowns?

Summary of the systematic method known as Gaussian elimination for solving systems of linear equations.

We write the linear system (LS) of m equations for the vector $\underline{\boldsymbol{x}} = [x_1, x_2, \ldots x_n]$ of the n unknowns as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad & \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

The matrix that we get by adjoining (augmenting) the right-side $\underline{\boldsymbol{b}}$-vector to the coefficient matrix $A = [a_{ij}]$ is called the *augmented matrix* $\langle A|\underline{\boldsymbol{b}}\rangle$:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right]$$

Our goal is to find all the solution vectors $\underline{\boldsymbol{x}}$ to the system - i.e. the *solution set* .

There are three types of *elementary equation operations* that don't change the solution set to the linear system. They are

- interchange two of equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

And that when working with the augmented matrix $\langle A|\underline{\boldsymbol{b}}\rangle$ these correspond to the three types of *elementary row operations:*

- interchange ("swap") two rows
- multiply one of the rows by a non-zero constant
- replace a row by its sum with a multiple of a different row.