Math 2270-3
Wednesday Sept. 9

- Tuesday notes on composition of matrix transformations, and matrix multiplication, §2.3

Then
§2.4 Inverse transformations and inverse matrices

<u>Inverse transformations and inverse matrix</u>

For $f(\vec{x}) = A_{m\times n}\vec{x}$, $f: \mathbb{R}^n \to \mathbb{R}^m$, when does there exist an inverse function
$g: \mathbb{R}^m \to \mathbb{R}^n$, i.e. so that $g(f(\vec{x})) = \vec{x} \ \forall \vec{x}$
$f(g(\vec{y})) = \vec{y} \ \forall \vec{y}$?

<u>Answer</u>: <u>f must be 1 to 1</u>: for each $\vec{y}$ in the range of f there must be exactly one $\vec{x} \in \mathbb{R}^n$ so that $f(\vec{x}) = \vec{y}$

& <u>f must be onto</u>: Every $\vec{y} \in \mathbb{R}^m$ must be $f(\vec{x})$ for some $\vec{x} \in \mathbb{R}^n$

These are the two conditions (in general) which are necessary and sufficient for an inverse function to exist.

$\boxed{\begin{array}{l} n>m: \ f \text{ is not 1-1} \\ \text{(more cols than rows)} \end{array}}$ Consider sol's to $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

$\downarrow$ rref

$$\left[\begin{array}{cccccc|c} 1 & * & & & & 0 & 0 \\ 0 & 0 & 1 & & & 0 & \vdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & \vdots \\ & & & & & & 0 \end{array}\right]$$

$n > m \Rightarrow \exists$ col. without leading 1 in rref
$\Rightarrow \infty$'ly many sol's to $A\vec{x} = \vec{0}$
$\therefore f$ not 1-1.

$\boxed{\begin{array}{ll} n<m & f \text{ is } \underline{\text{not}} \text{ onto!} \\ \text{(more rows than cols)} & \end{array}}$

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & ? \\ a_{21} & \cdots & a_{2n} & ? \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & ? \end{array}\right]$$

$\downarrow$ rref

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & ? \\ 0 & 1 & \cdots & 0 & \vdots \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & ? \end{array}\right]$$

at least one zero row in rref(A).
If augment with ? vector having a "1" in last entry & reverse row ops, → work backwards to an inconsistent system $A\vec{x} = \vec{b}$.
i.e. $\vec{b}$ is not in range(f)!

So, the only chance for a matrix transformation $f: \mathbb{R}^n \to \mathbb{R}^m$ $f(\vec{x}) = A\vec{x}$ to have an inverse function is when $n=m$, i.e. $A_{n\times n}$ is square.

So, if $A_{n\times n}$ is square, what conditions guarantee that $\forall \vec{y} \in \mathbb{R}^n$, $A\vec{x} = \vec{y}$ has a unique sol'n $\vec{x}$?
i.e. f is 1-1 & onto

Try to solve $A\vec{x} = \vec{y}$:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & y_1 \\ a_{21} & & & & y_2 \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & & a_{nn} & y_n \end{array}\right]$$

$\swarrow$ $\mathrm{rref}(A) = I$

$$\left[\begin{array}{ccccc|c} 1 & 0 & \cdots & & 0 & z_1 \\ 0 & 1 & \cdots & & 0 & z_2 \\ 0 & 0 & 1 & \cdots & 0 & z_3 \\ & & \vdots & & & \\ 0 & 0 & \cdots & & 1 & z_n \end{array}\right]$$

notice there is a unique solution $\vec{x} = \vec{z}$ in this case, and the formulas for each ~~component~~ component z_i are linear combos of the $y_1, \dots, y_n$, so

$\vec{z} = B\vec{y}$ for some matrix B!

Inverse exists and is a matrix transformation!

$\searrow$ $\mathrm{rref}(A) \neq I$

$$\left[\begin{array}{cccccc|c} 1 & 0 & \cdots & & & 0 & z_1 \\ 0 & 1 & \cdots & & & 0 & z_2 \\ 0 & 0 & 0 & 1 & \cdots & 0 & \\ & & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & & 0 & z_n \end{array}\right]$$

~~$\$$~~ rref(A) has at least bottom row of zeroes, and at least one col without a leading 1.
So f is neither 1-1 nor onto.

No inverse exists!

We write A^{-1} for B.

Since

$$g(f(\vec{x})) = \vec{x}, \quad B(A(\vec{x})) = (BA)\vec{x} = I\vec{x} \quad \forall \vec{x}$$
$$f(g(\vec{y})) = \vec{y}, \quad A(B(\vec{y})) = (AB)\vec{y} = I\vec{y} \quad \forall \vec{y}$$

so $BA = I$
$\quad AB = I$

How to find A^{-1}, which exists iff $\mathrm{rref}(A) = I$.

$$A \left[\, col_1(B) \; \vdots \; col_2(B) \; \vdots \; \cdots \; col_n(B) \, \right] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & \vdots & 1 & & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

find all n col's at once synthetically!

$$\left[\; A \; \middle| \; \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & 0 & & \vdots \\ 0 & 0 & & 1 \end{matrix} \right]$$

$$\Big\downarrow \text{rref}$$

$$\left[\; I \; \middle| \; \underbrace{col_1(B) \; \vdots \; col_2(B) \; \vdots \; \cdots col_n(B)}_{B\,!} \right]$$

<u>example</u>: Find $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}^{-1}$, and check!

$$\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ \hline \end{array}$$

also works 3x3 etc:
Find $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix}^{-1}$
and check!

$$\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$