Math 2270-3
Friday Sept. 4.
§2.2
Monday is Labor Day
– no class!

HW for Friday Sept. 11

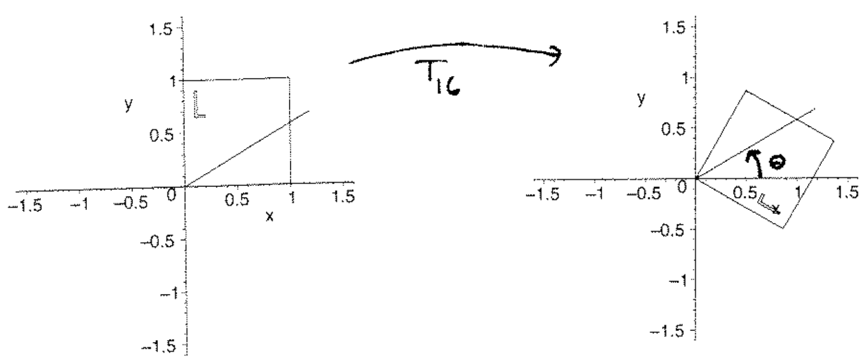
2.1 8, 10, 13, 16, 17, 19, 24, 25, 26, 28
30, 32, 36, 42
2.2 1 2 (on 1, 2 draw our L-box images instead of Bretscher's L), 9, 10, 11, 12, 13, 17 27
35, 36, 38 43 44 49
2.3 1, 2, 5, 6, 16 29
2.4 1, 4, 14

Exercises "4", "5" in today's notes

- Finish Wednesday's notes
  – this will take ~~po~~ most of the lecture.

- Discuss how to find the inverses to linear transformations, where applicable. (This will tie in to inverse matrices in §2.4, once we discuss matrix multiplication in §2.3 .... for now we'll proceed like you used to in algebra.)

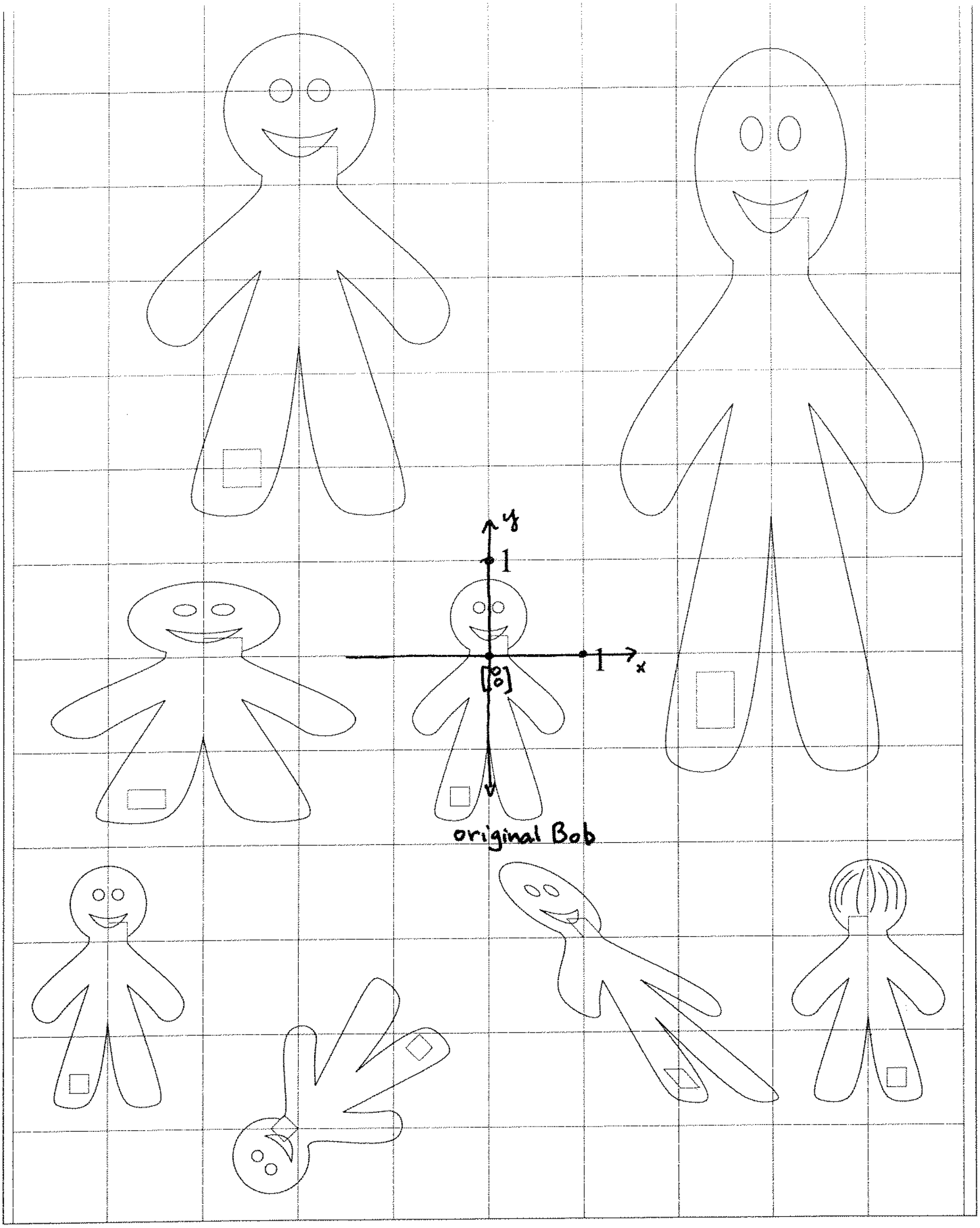
I forgot (at least) one example! reflection through arbitrary lines!



reflect thru a line thru origin at angle θ, with unit direction
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

A slight generalization of linear transformation is <u>affine transformation</u>

An affine transformation $a(\vec{x}) = A\vec{x} + \vec{b}$ is the composition of a matrix (linear) transformation, followed by a translation. Can you figure out the formulas for the affine transformations below?

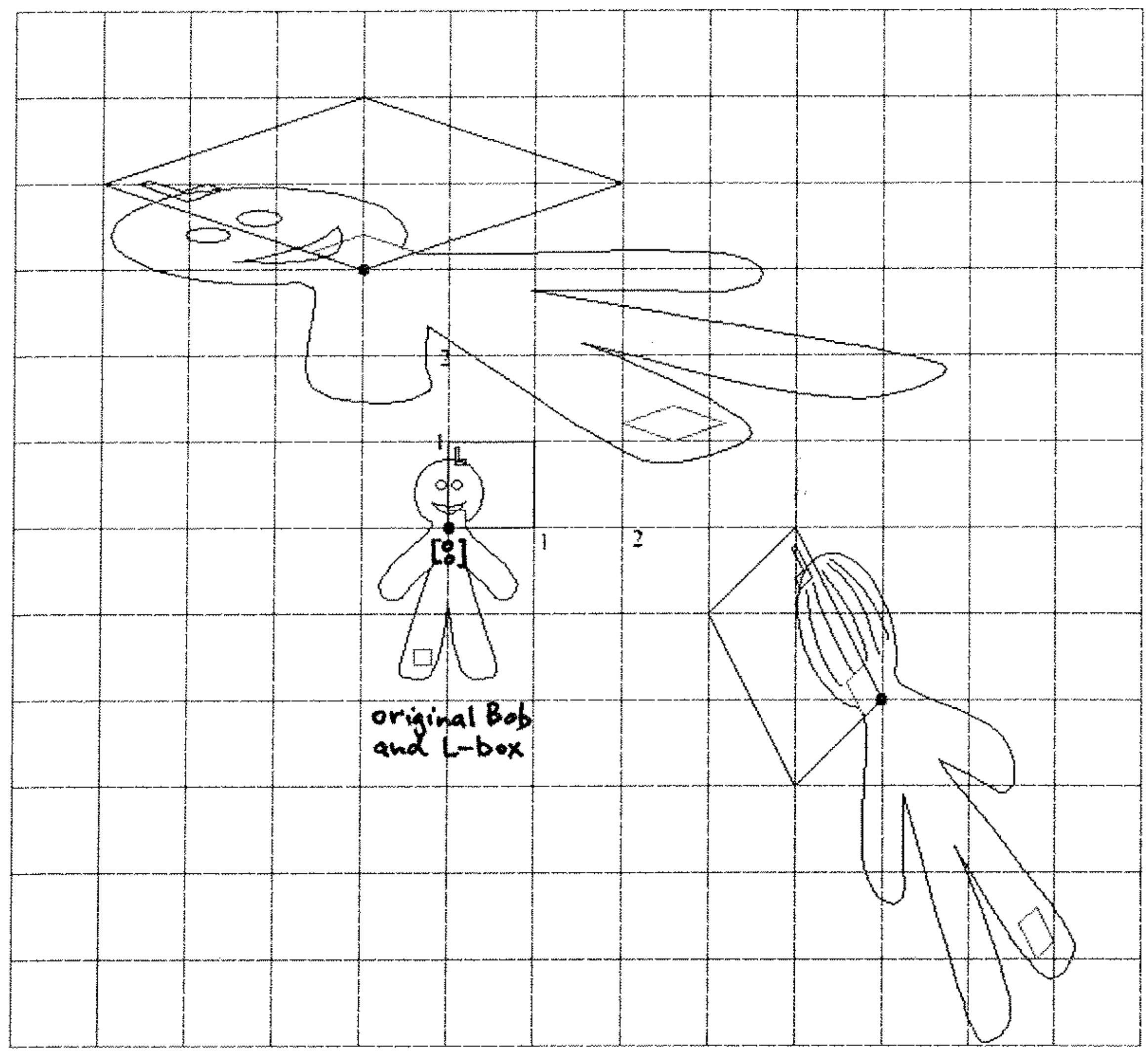


Homework exercises:

Bob and L-box transform themselves:

Exercise 4. Find formulas for the two affine maps which are shown.

Exercise 5. Show where the L-box is transformed to by $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \end{bmatrix}$. If you want, you can draw in a piece of transformed Bob.



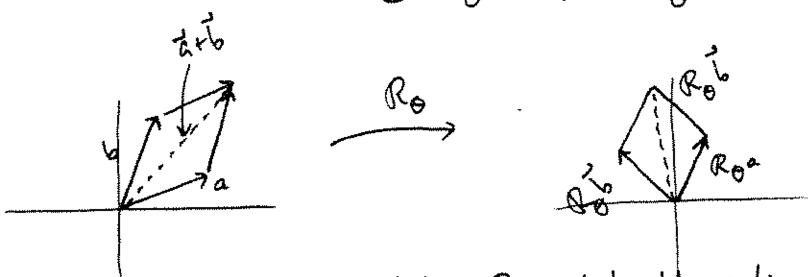
We will use these ideas soon, when we study (iterated function system) fractals.
See http://www.math.utah.edu/~korevaar/fractals.

Linear and affine transformation details: $T: \mathbb{R}^2 \to \mathbb{R}^2$ linear, or $\mathcal{A}: \mathbb{R}^2 \to \mathbb{R}^2$ affine.

<u>Rotations</u> are linear, because ① $R_\theta(\vec{a}+\vec{b}) = R_\theta(\vec{a}) + R_\theta(\vec{b})$

② $R_\theta(k\vec{a}) = k\,R_\theta(\vec{a})$

①

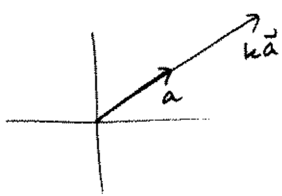


the rotation R_θ rotates the entire $\vec{a}, \vec{b}$ parallelegram, including the diagonal.

Thus, looking at the image on the right,

$$R_\theta(\vec{a}+\vec{b}) = R_\theta(\vec{a}) + R_\theta(\vec{b})$$

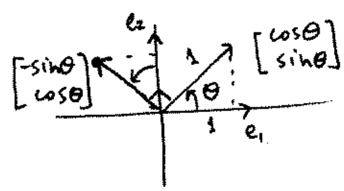
②



the rotation R_θ rotates $\vec{a}, k\vec{a}$ to $R_\theta\vec{a}$ and $k\,R_\theta\vec{a}$.

Since rotation is linear its matrix is given by.

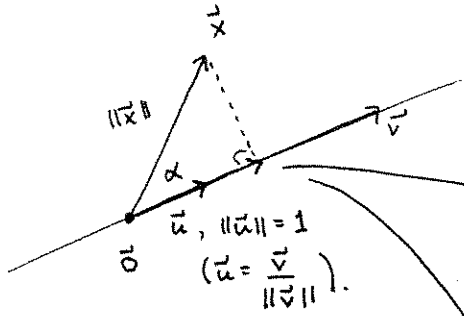
$$[R_\theta] = \left[\begin{array}{c|c} R_\theta\vec{e}_1 & R_\theta\vec{e}_2 \end{array}\right]$$



$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

So $$\boxed{R_\theta\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}.}$$

projections and reflections



$\text{comp}_{\vec{u}}\vec{x} = \|x\|\cos\alpha$ — scalar component of $\vec{x}$ in direction of $\vec{u}$

$= \vec{x}\cdot\vec{u}$ $\quad \left(= \vec{x}\cdot\left(\frac{\vec{v}}{\|\vec{v}\|}\right)\right)$

since $\vec{x}\cdot\vec{u} = \|\vec{x}\|\underbrace{\|\vec{u}\|}_{=1}\cos\alpha$

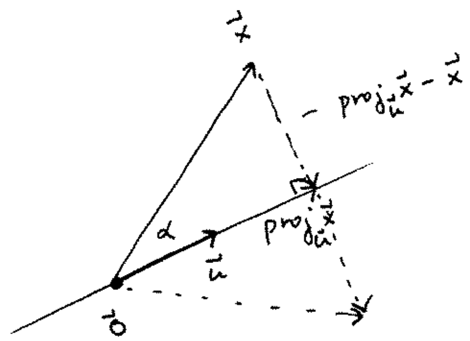
$\text{proj}_{\vec{u}}\vec{x}$ — vector projection of $\vec{x}$ onto $\vec{u}$

$$= \underbrace{(\vec{x}\cdot\vec{u})}_{\text{how far to go}}\ \underbrace{\vec{u}}_{\text{the unit direction.}}$$

projection

$$\text{so, } \text{proj}_{\vec{u}}\vec{x} = (x_1u_1 + x_2u_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1^2x_1 + u_1u_2x_2 \\ u_1u_2x_1 + u_2^2x_2 \end{bmatrix} = \boxed{\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \text{proj}_{\vec{u}}\vec{x}}$$

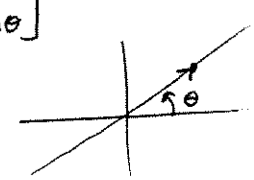
reflection:



$$\begin{aligned} \text{refl}_{\vec{u}}\vec{x} &= \vec{x} + 2(\text{proj}_{\vec{u}}\vec{x} - \vec{x}) \\ &= -\vec{x} + 2\,\text{proj}_{\vec{u}}\vec{x} \\ &= \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} + 2\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -1+2u_1^2 & 2u_1u_2 \\ 2u_1u_2 & -1+2u_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

if we write

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, with trig identities we see



$$\boxed{\text{refl}_{\vec{u}}\vec{x} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

!!