Math 2270-3
Tuesday Sept. 29
§3.4 cont'd

- do page 5 Monday: studying how, for some linear transformations, bases other than the standard basis are best.

Recall

$\beta = \{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_n\}$ basis for $\mathbb{R}^n$.

$T(\vec{x}) = A\vec{x}$ linear

$S = \left[\vec{w}_1 \middle| \vec{w}_2 \middle| \cdots \middle| \vec{w}_n\right]$

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] & T(\vec{x}) \arrow[d, dashed, "S^{-1}"] \\ {[\vec{x}]_\beta} \arrow[u, "S"] \arrow[r, "B"] & {[T(\vec{x})]_\beta} \end{tikzcd}$$

$\mathrm{col}_j(B) = [T(\vec{w}_j)]_\beta$

and

so $B = S^{-1}AS$

This motivates:

<u>Definition</u> A and B are <u>similar matrices</u> iff $\exists$ invertible S so that

$$B = S^{-1}AS$$

<u>example</u> $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

<u>notice</u>: If A and B are similar you can think of A as the matrix of $f(\vec{x}) = A\vec{x}$ with respect to the standard basis, and B as the matrix of the <u>same</u> transformation function, but with respect to basis given by the columns of S. } or vise versa!
Thus many <u>matrix</u> properties of A & B will turn out to be "similar"

<u>notice</u>: if $B = S^{-1}AS$ then $SBS^{-1} = A$ $\left[\,(S^{-1})^{-1}BS\,\right]$ so being similar is <u>symmetric</u> in A & B

also, if $B = S^{-1}AS$ and $C = T^{-1}BT$
then $C = T^{-1}S^{-1}AST = (ST)^{-1}A(ST)$ so C is similar to A, i.e.

being similar is a <u>transitive</u> property

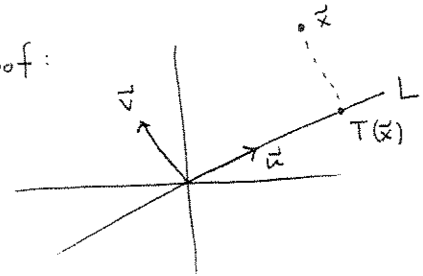
and $A = I^{-1}AI$ so A is similar to A;
being similar is <u>reflexive</u>

Any relationship between objects which is <u>reflexive</u>, <u>symmetric</u>, and <u>transitive</u> is called an <u>equivalence relation</u>, and allows you to partition your objects into subsets (equivalence classes) consisting of all mutually equivalent objects.

Examples of similar matrices for $T : \mathbb{R}^2 \to \mathbb{R}^2$:

1. Every $\mathbb{R}^2$ projection matrix A is similar to $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

proof:



let $T(\vec{x})$ be projection onto L

pick
- $\vec{u}$ a (unit) direction vector for L
- $\vec{v}$ a (unit) $\perp$ vector to L

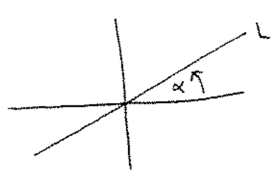
Let $\mathcal{B} = \{\vec{u}, \vec{v}\}$.

Thus
$$T(\vec{u}) = 1\vec{u} + 0\vec{v}$$
$$T(\vec{v}) = 0\vec{u} + 0\vec{v}$$

So
$$[T]_{\mathcal{B}} = \left[[T(\vec{u})]_{\mathcal{B}} \;\vdots\; [T(\vec{v})]_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= B = \underset{\downarrow}{S^{-1}}AS$$
$$[\vec{u} | \vec{v}].$$

2. What simple matrix is every reflection matrix similar to?

3. Use similar matrices to show that if L makes an angle α with $\vec{e}_1$, then the matrix for reflection across L is (w.r.t. standard basis!)



$$A = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} \quad !$$

Hint: If $B = S^{-1}AS$
then $SBS^{-1} = A$!

(4) Show $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ is similar to $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and determine a, b.

In order for this to be true, then for $T(\vec{x}) := A\vec{x}$
there must be a basis $\beta = \{\vec{u}, \vec{v}\}$ so
that $T\vec{u} = a\vec{u} + 0\vec{v}$ ($col_1(B)$ coeff's)
$T\vec{v} = 0\vec{u} + b\vec{v}$ (coeffs in $col_2(B)$.)

So we search for vectors $\vec{w}$ satisfying

$A\vec{w} = \lambda\vec{w}$ ($\lambda = a, b$ ultimately)

note: if such $\vec{w} \neq \vec{0}$ exist they're called eigenvectors of A, with eigenvalues λ.

iff $A\vec{w} - \lambda\vec{w} = \vec{0}$
iff $A\vec{w} - \lambda I\vec{w} = \vec{0}$
$(A - \lambda I)\vec{w} = \vec{0}$.

So $A - \lambda I$ needs to have non-zero kernel
So $A - \lambda I$ needs to not have an inverse (not rref I, etc.)
So $\det(A - \lambda I) = 0$ needs to be true

$$\det\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = (\lambda-1)(\lambda-3) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1) = 0$$

so $\lambda = 5, -1$ (these are our a, b values!) or viseverse.

$\lambda = 5$:

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \\ \hline 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$w_1 = t$
$w_2 = 2t$ $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t$

$\lambda = -1$:

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$w_1 = -t$
$w_2 = t$ $\vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} s$

Thus, for $S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

$$S^{-1}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}[S] = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

5 An interesting fact is that any $A_{2\times 2}$ is similar to exactly one of the following matrices:

(i) $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ includes uniform & non-uniform scaling, projection, reflection, dilation projections & reflections

(ii) $B = c\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ uniform scaling composed with a shear (and possibly reflection thru the origin, i.e. $c<0$)

(iii) $B = c\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ rotation dilation

we'll be able to check this when we return to the study of eigenvalues & eigenvectors later in the course.

(If you look at page 3, you'll notice case (i) happens when we can find two linearly independent eigenvectors. Case (ii) happens when theres only 1 eigenvalue (1 real double root), and only one linearly ind. eigenvector. Case (iii) is what happens when the eigenvalues are complex #'s!)