Hw for Friday Oct 2

3.4 1 3 5 6, 11, 14, 15 16, 17, 21, 22, 31 33, 43, 53, 56

4.1 1, 2, 5, 6, 10, 13, 14, 20, 25, 30, 35, 48

Math 2270-3
Monday Sept 28

§3.4: coordinates; matrices of linear transformations with respect to different bases.

One reason a linearly independent collection of spanning vectors for a subspace (i.e. a basis) is a good thing for a subspace is this fact:

Theorem Let $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k\}$ be a basis for the subspace $W \subset \mathbb{R}^n$.
Then each $\vec{w} \in W$ is a unique linear combination of the basis vectors,
i.e.

$$\vec{w} = c_1 \vec{w}_1 + c_2 \vec{w}_2 + \cdots + c_k \vec{w}_k$$

for unique scalars $c_1, c_2, \ldots c_k$.

Definition: In this case we call the vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$ the coordinates of $\vec{w}$ with respect to the basis
$\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k\}$, and write $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$.

proof of Thm: Since $\mathcal{B} := \{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k\}$ is a basis for W, then we can write

$$\vec{w} = c_1 \vec{w}_1 + c_2 \vec{w}_2 + \cdots + c_k \vec{w}_k$$ for at least one collection of lin-combo coeffs. (since bases span W).

If also $\vec{w} = d_1 \vec{w}_1 + d_2 \vec{w}_2 + \cdots + d_k \vec{w}_k$
then $\vec{0} = \vec{w} - \vec{w} = (c_1 - d_1)\vec{w}_1 + (c_2 - d_2)\vec{w}_2 + \cdots (c_k - d_k)\vec{w}_k$.

By linear independence $\left.\begin{matrix} c_1 - d_1 = 0 \\ c_2 - d_2 = 0 \\ \vdots \\ c_k - d_k = 0 \end{matrix}\right\}$ so $c_1 = d_1, c_2 = d_2, \ldots c_k = d_k$ ∎

Example Let $W = \mathbb{R}^n$. Let $\{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\} := \mathcal{E}$ be the standard basis. $\left(\text{entry}_i(\vec{e}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}\right)$

then $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n$

and $[\vec{x}]_{\mathcal{E}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$!

Example : Even for $\mathbb{R}^2$ there are times when the standard basis is not ideal.

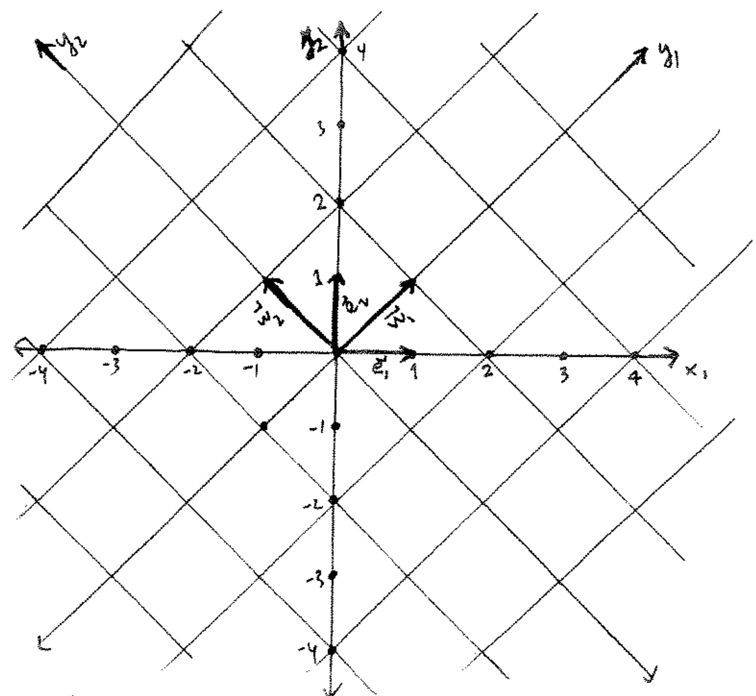
Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$

(a) Find $[\vec{x}]_{\mathcal{B}}$, and interpret geometrically, for

$\vec{x} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Hint : If we write $[\vec{x}]_{\mathcal{B}} = \vec{y}$, then

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



(b) Find $\vec{x} = [\vec{x}]_{\mathcal{E}}$ if $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

interpret geometrically (i.e. sketch).

(c) Consider the curve in $\mathbb{R}^2$ given implicitly by $x_1 x_2 = 1$. What is its equation in the y_1-y_2 coordinate system?

Hint: $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Sketch the curve!

<u>Example</u> Let W be the plane $x_1 + 2x_2 + 3x_3 = 0$ in $\mathbb{R}^3$.

Let $\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ be a basis for W.

Find $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} \in W$.

The matrix of a linear transformation, with respect to a given basis

We've seen that if $\{\vec{w}_1, \vec{w}_2, .. \vec{w}_k\}$ is a basis for subspace $W < \mathbb{R}^n$
and if $\{\vec{e}_1, \vec{e}_2, .. \vec{e}_n\}$ is the standard basis for $\mathbb{R}^n$

Then for $\vec{w} = c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_k\vec{w}_k \in W$

$$\vec{w} = \underbrace{\begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix}}_{\text{"S"}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}}_{[\vec{w}]_\beta}.$$

$$[\vec{w}]_{\mathcal{E}} = \vec{w} = S[\vec{w}]_\beta$$

If $W = \mathbb{R}^n$, then $S_{n\times n}$ is invertible and $\vec{w} = S[\vec{w}]_\beta$;
$S^{-1}\vec{w} = [\vec{w}]_\beta$.

Now let $T: \mathbb{R}^n \to \mathbb{R}^n$ linear, $T(\vec{x}) = A\vec{x}$.

<u>Definition</u> Let $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, .. \vec{w}_n\}$ be a basis for $\mathbb{R}^n$, $T(\vec{x}) = A_{n\times n}\vec{x}$ as above.
Then the <u>matrix for T</u> <u>with respect to $\mathcal{B}$</u>, $[T]_\beta$, is the matrix B which satisfies

$$B[\vec{x}]_\beta = [T(\vec{x})]_\beta.$$

In other words, $[T]_\beta$ shows how β-coordinates transform when T is applied.

The following diagram shows that $[T]_\beta = B$ exists, and one way to compute it: (edges are labeled with the matrices of corresponding linear transformations)

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, dotted, bend right, "S^{-1}"'] & T(\vec{x}) \arrow[d, "S^{-1}"] \\
{[\vec{x}]_\beta} \arrow[u, "S"'] \arrow[r, "B"] & {[T(\vec{x})]_\beta} \arrow[u, dotted, bend left, "S"]
\end{tikzcd}$$

$$B[\vec{x}]_\beta = \underbrace{\left(S^{-1}\underbrace{\left(A\underbrace{\left(S[\vec{x}]_\beta\right)}_{\vec{x}}\right)}_{T(\vec{x})}\right)}_{[T(\vec{x})]_\beta}$$

So $\boxed{B = S^{-1}AS}$ works. Method 1

Letting $\vec{x} = \vec{w}_j$,
$[\vec{x}]_\beta = \vec{e}_j$, deduce $col_j(B) = col_j(S^{-1}AS)$ so
B is unique.

Alternate way to compute B:

If $\vec{x} = \vec{w}_j$, $[\vec{w}_j]_\beta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow$ entry j.

So by def. of what B does,

$$col_j(B) = B[\vec{w}_j]_\beta = [T(\vec{w}_j)]_\beta$$

$$\boxed{col_j(B) = [T(\vec{w}_j)]_\beta}$$

Method 2

Example

$\mathcal{B} = \{\vec{w}_1, \vec{w}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

(same $\mathbb{R}^2$ basis as page 2.)

$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (this transformation is better understood using β!)

Method 1

$B = S^{-1} A S$

Fill in both methods!

Method 2

$col_1(B) = [T(\vec{w}_1)]_\beta$; $col_2(B) = [T(\vec{w}_2)]_\beta$

This transformation $\mathbf{T}(\vec{x})$ is better understood using $\mathcal{B}$!