Math 2270-3
Tuesday Sept. 22

* remind me to explain why rref(A) is unique – using big Maple example

§3.3 bases and dimension, for subspaces.

We'll be using definitions and ideas from the past several lectures...
What do the following words mean?

Subspace $W \subset \mathbb{R}^n$

linear combination of $\vec{v}_1, \vec{v}_2, .. \vec{v}_k$

span of $\{\vec{v}_1, \vec{v}_2, - \vec{v}_k\}$

for $T(\vec{x}) = A\vec{x}$; $T: \mathbb{R}^n \to \mathbb{R}^m$ linear

ker T

Image T

$\{\vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$ linearly dependent

$\{\vec{v}_1, .. \vec{v}_k\}$ linearly independent

$\{\vec{v}_1, .. \vec{v}_k\}$ a basis for the subspace W.

dimension of W

- Yesterday we showed that if a collection of vectors spans a subspace W, we can successively remove ones which are in the span of those that remain, until we finally cull the original set into a basis for W. We did this procedure to the columns of A to obtain a basis for Image(T) consisting of some of the original columns

- A complementary procedure, which we used when we found all subspaces of $\mathbb{R}^3$, is to start with an independent collection of vectors $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_\ell\}$ in W. If this set does not yet span W, find

$$\vec{w} \in W; \quad \vec{w} \notin \text{span}\{\vec{v}_1, \vec{v}_2, .. \vec{v}_\ell\}.$$

Then $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_\ell, \vec{w}\}$ is also independent. (In this way you can build up to a basis!)

proof: Let $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_\ell\vec{v}_\ell + d\vec{w} = \vec{0}$

Case I: $d \neq 0$. Then $\vec{w} = -\frac{c_1}{d}\vec{v}_1 - \frac{c_2}{d}\vec{v}_2 - \ldots - \frac{c_\ell}{d}\vec{v}_\ell$

so this case did not happen!

Case II: $d = 0$. Then $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_\ell\vec{v}_\ell = \vec{0}$

Since $\{\vec{v}_1, \vec{v}_2, - \vec{v}_\ell\}$ lin. ind.,

each $c_1 = c_2 = .. = c_\ell = 0$

since also $d = 0$, deduce

$\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_\ell, \vec{w}\}$ lin independent ■

Today we will prove & learn important facts about bases for $\mathbb{R}^n$ and for subspaces. Almost every one of these facts uses rref.

1. Every basis of $\mathbb{R}^n$ must have exactly n vectors, because

   - more than n vectors must be dependent

     e.g. could $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$ be independent?

   - fewer than n vectors cannot span $\mathbb{R}^n$

     e.g. could $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 27 \\ 36 \\ 2 \end{bmatrix} \right\}$ span $\mathbb{R}^3$?

2. $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ iff $\text{rref}\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \middle| \vec{v}_n \right] = I$.

3 Now let W be any subspace of $\mathbb{R}^n$.

- We can find a basis $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ for W by successively adding vectors until we have an independent spanning set, just like we did in our discussion of $\mathbb{R}^3$ subspaces... the process must stop for $k \le n$, since $>n$ vectors in $\mathbb{R}^n$ are dependent

Then we deduce

(a) If $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_\ell\}$ are vectors in W with $\ell > k$, then this set <u>must be dependent.</u>

(proving this fact only uses that $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ <u>span</u> W)

Hint: You can write

$$n\left\{ \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix}}_{k} \right. \begin{bmatrix} & A & \end{bmatrix}_{k\times \ell} = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_\ell \end{bmatrix}_{n\times \ell}$$

where the j^{th} col of A expresses $\vec{w}_j$ as a linear combo of $\{\vec{v}_1, \dots \vec{v}_k\}$.

(b) If $\{\vec{w}_1, \dots \vec{w}_\ell\} \subset W$ with $\ell < k$, then this set <u>cannot span W</u>

(Use part (a), reverse roles & use logic!)

Notice, (a), (b) prove that <u>every</u> basis of W has k vectors, so the concept of dimension for a subspace is well-defined (doesn't depend on which basis you use.)

(c) If $\dim(W) = k$ (as it does here), then

if $\{\vec{w}_1, \vec{w}_2, .. \vec{w}_k\}$ span W, they are automatically independent $\Rightarrow$ basis!

<u>proof</u>: If dependent, remove vectors to get a basis with $< k$ vectors. impossible!

(d) If $\dim(W) = k$ and if $\{\vec{w}_1, \vec{w}_2, .. \vec{w}_k\} \subset W$ are linearly ind., then they automatically span W, $\Rightarrow$ basis!

<u>proof</u>: If $\{\vec{w}_1, .. \vec{w}_k\}$ doesn't span, successively add vectors to get a basis with $> k$ vectors. impossible!

<u>examples</u>: • Are $\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} \right\}$ a basis for the plane $x + 2y - 3z = 0$?

• Find a basis for $\mathbb{R}^3$ including $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

4 (rank + nullity Theorem)

If $f(\vec{x}) = A\vec{x}$

$f: \mathbb{R}^n \to \mathbb{R}^m$

$$\underbrace{\dim(\ker f)}_{\text{"nullity"}} + \underbrace{\dim(\text{image}(f))}_{\text{"rank"}} = \dim(\text{domain}) = n$$

$\Updownarrow$

before, we said rank(A) = # of non-zero rows in rref(A).
why is this the same as $\dim(\text{image}(f))$?

Examples
- the Maple example from yesterday $T(\vec{x}) = \begin{bmatrix} & A_{4\times 6} & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$

- $T: \mathbb{R}^n \to \mathbb{R}^n$ $T(\vec{x}) = A_{n\times n}$; A^{-1} exists:

- $T: \mathbb{R}^n \to \mathbb{R}^n$ $T(\vec{x}) = \vec{0}$ $\forall \vec{x}$.