Math 2270-3
Monday Sept 21

We will go through the large (Maple) example from Friday (Wed) last week.
We will use the definitions and concepts below - you need to memorize and learn how to use these!

<u>Def</u>: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
Then

$$\text{Image}(T) := \{ \vec{y} \in \mathbb{R}^m \text{ s.t. } \exists\ \vec{x} \in \mathbb{R}^n \text{ with } T(\vec{x}) = \vec{y} \}$$
$$\text{kernel}(T) := \{ \vec{x} \in \mathbb{R}^n \text{ s.t. } T(\vec{x}) = \vec{0} \}.$$

<u>Def</u>: A <u>linear combination</u> of the vectors $\vec{v}_1, \vec{v}_2, .. \vec{v}_k$ is any sum of scalar multiples of these vectors, i.e. any $\vec{v}$ expressable as

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k \qquad c_1, c_2, .. c_k \in \mathbb{R}.$$

<u>Def</u>: <u>span</u> $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$ is the collection of all linear combinations, i.e.

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\} = \left\{ \vec{v} \text{ s.t. } \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k \text{ some } \atop c_1, c_2 .. c_k \in \mathbb{R} \right\}$$

<u>Def</u> $W \subset \mathbb{R}^n$ is a <u>subspace</u> iff W is closed under addition and scalar multiplication, i.e. iff whenever $\vec{u}, \vec{v} \in W$, $k \in \mathbb{R}$, then

(i) $\vec{u} + \vec{v} \in W$
(ii) $k\vec{u} \in W$

<u>Examples</u>
- Let $T: \mathbb{R}^n \to \mathbb{R}^m$ linear. Then $\ker(T)$, $\text{Im}(T)$ are subspaces.
- span $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$ is a subspace.

<u>Def</u> $\vec{v}$ is <u>linearly dependent</u> ~~of~~ on $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$ if $\vec{v}$ is in span $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$,
i.e. $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$

$\vec{v}$ is linearly <u>independent</u> of $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$ if it's <u>not</u> in their span

<u>Definition</u> $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is <u>linearly dependent</u> if at least one of the $\vec{v}_j$ is dependent on the remaining vectors. Equivalently, some linearly combination

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0} \qquad \text{with not all of the } c_\ell = 0.$$

<u>Definition</u> $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is <u>linearly independent</u> if no $\vec{v}_j$ is a linear combo of the rest. Equivalently, if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}, \quad \text{then } c_1 = c_2 = \dots = c_k = 0.$$

<u>Definition</u>: If $W = \text{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ and $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is linearly independent, then $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is called a <u>basis</u> for W. And, we define the dimension of W, <u>dim(W)</u> to be the number of vectors, <u>k</u>, in this (and it turns out, any) basis for W

<u>Basic Example</u>: Check that the "standard basis", $\{\vec{e}_1, \vec{e}_2, \dots \vec{e}_n\}$ is a basis for $\mathbb{R}^n$. (So $\mathbb{R}^n$ is n-dim'l!)

a): linearly independent:

b) span:

In our last several lectures we make use of these two important facts:

① <u>removing redundant vectors</u>: If $\vec{w}$ is dependent on $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$, then $\text{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k, \vec{w}\} = \text{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$

<u>proof</u>: if $\vec{w} = d_1\vec{v}_1 + d_2\vec{v}_2 + \dots + d_k\vec{v}_k$
then $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k + d\vec{w} = (c_1 + dd_1)\vec{v}_1 + (c_2 + dd_2\vec{v}_2) + \dots + (c_k + dd_k)\vec{v}_k$ ∎

② <u>adding vectors not in the span</u>: If $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ are lin. ind. and $\vec{w}$ is <u>not</u> in $\text{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ Then $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k, \vec{w}\}$ is lin. ind.

<u>proof</u>: Let $c_1\vec{v}_1 + \dots + c_k\vec{v}_k + d\vec{w} = \vec{0}$

<u>Case I</u>: $d \neq 0$. Then $\vec{w} = -\frac{c_1}{d}\vec{v}_1 - \frac{c_2}{d}\vec{v}_2 - \dots - \frac{c_k}{d}\vec{v}_k$.
Since $\vec{w} \notin \text{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ this case cannot occur.

<u>Case II</u>: $d = 0$. Then $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$
Thus $c_1 = c_2 = \dots = c_k = 0 = d$ ∎