Math 2270-3
Wednesday Sept. 2
2.1-2.2 : Matrix transformations $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n \to \mathbb{R}^m$
$\mathbb{R}^2 \to \mathbb{R}^2$

- Finish pages 3-4-5 Tuesday, especially example page 5.

- Some general thoughts before beginning §2.2 examples

In $\mathbb{R}^n$ we write $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, ... $\vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$, i.e. $\text{entry}_i(\vec{e}_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

We call $\{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$ the <u>standard basis vectors</u> for $\mathbb{R}^n$,
because each $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is easily and uniquely expressable as the linear combination

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n.$$

<u>example</u> $\begin{bmatrix} 3 \\ 7 \\ -6 \end{bmatrix} = 3\vec{e}_1 + 7\vec{e}_2 - 6\vec{e}_3$

<u>example</u> For $T(\vec{x}) = A\vec{x}$, $A\vec{e}_j = \text{col}_j(A)$, the j^{th} column.

Converse to Theorem on page 4 Wed:

<u>Theorem</u>

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be any transformation which satisfies

$$\left.\begin{array}{ll} \text{(a)}\ T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) & \forall\, \vec{u}, \vec{v} \in \mathbb{R}^n \\ \text{(b)}\ T(k\vec{u}) = kT(\vec{u}) & \forall\, \vec{u} \in \mathbb{R}^n, k \in \mathbb{R} \end{array}\right\} \text{These are called the linearity axioms}$$

Then T is actually a matrix transformation.

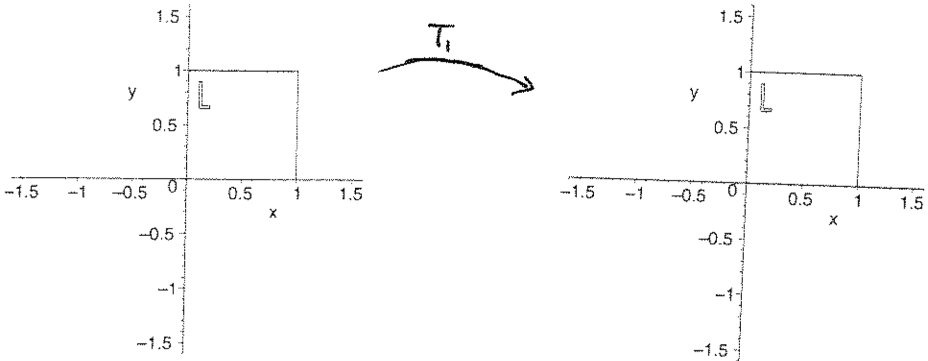
<u>proof</u>:
$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) &= T\left(x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n\right) \\ &= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \ldots + T(x_n\vec{e}_n) && \text{repeated application of (a)} \\ &= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \ldots + x_n T(\vec{e}_n) && \text{(b)} \\ &= \left[\, T(\vec{e}_1) \,\vdots\, T(\vec{e}_2) \,\vdots\, \cdots \,\vdots\, T(\vec{e}_n) \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\vec{x} && \text{linear combo form of matrix mult.} \end{aligned}$$

∎

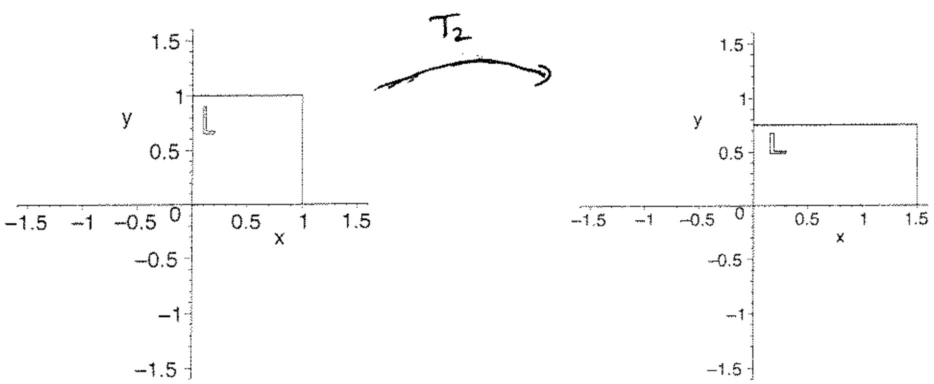
§2.2 Important linear transformations $T: \mathbb{R}^2 \to \mathbb{R}^2$

Bretscher likes to use an "L" to represent the geometry of the transformation.
I like to use "L boxes" instead ~ will tie into Maple project 1 on fractals.

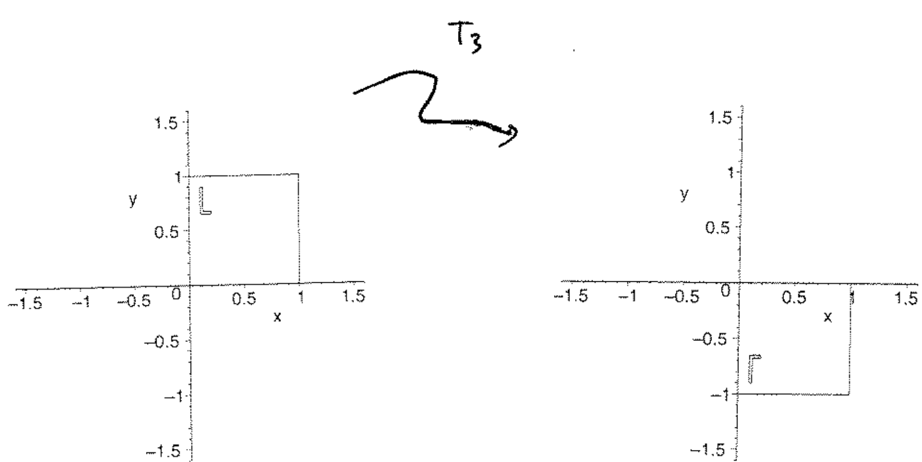
Find the matrix formulas for these transformations!



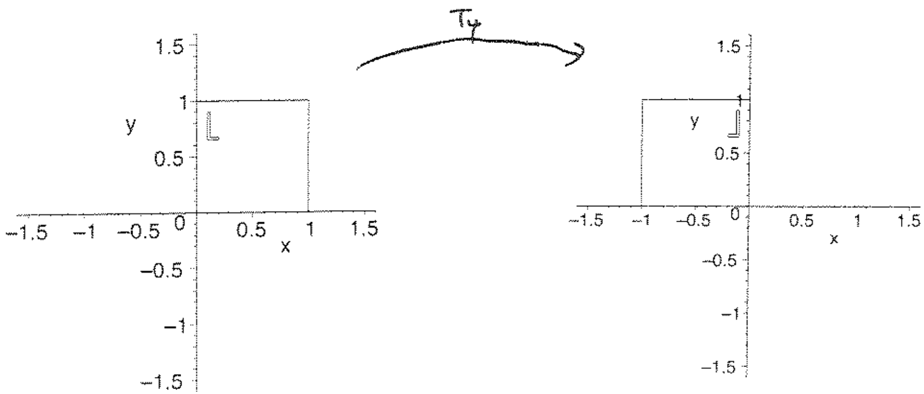
identity transformation



stretch by 1.5 horizontally & .75 vertically (from origin)

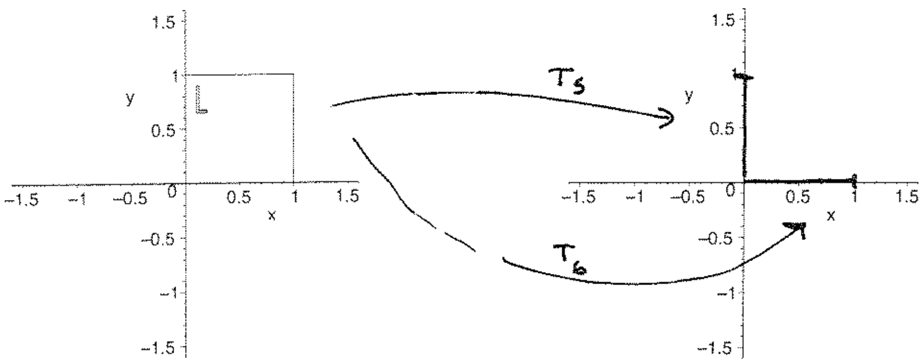


reflect across x-axis

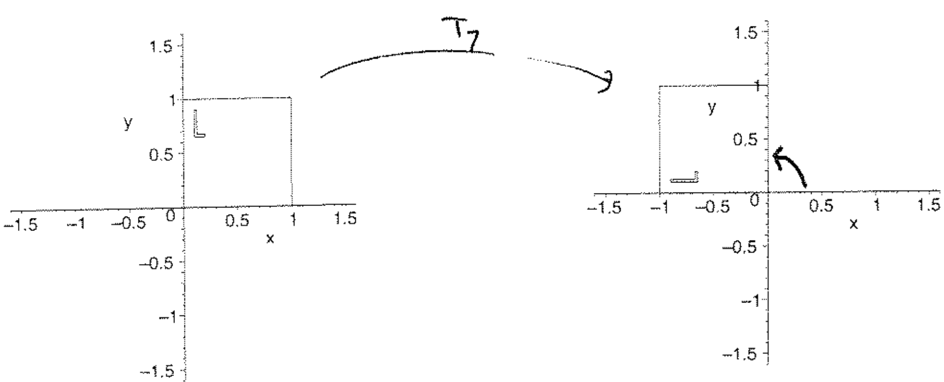


reflect across y-axis

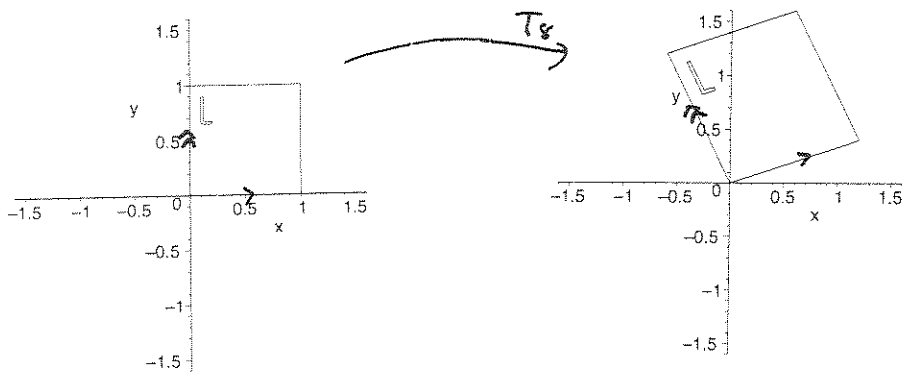
project to y-axis



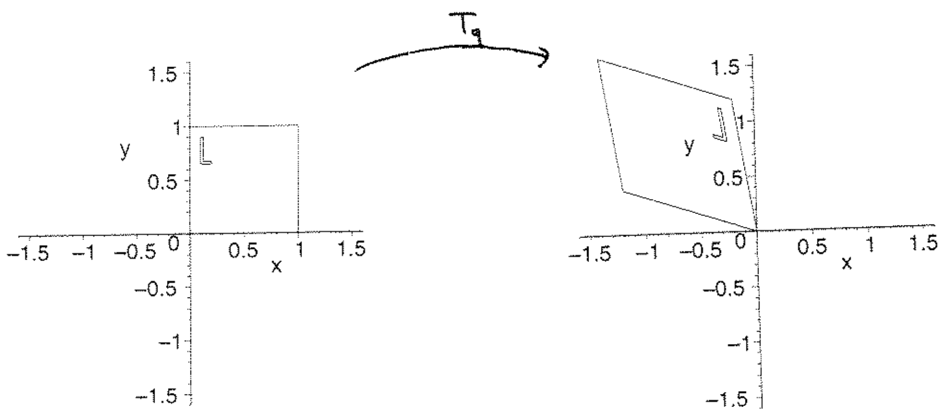
project to x-axis



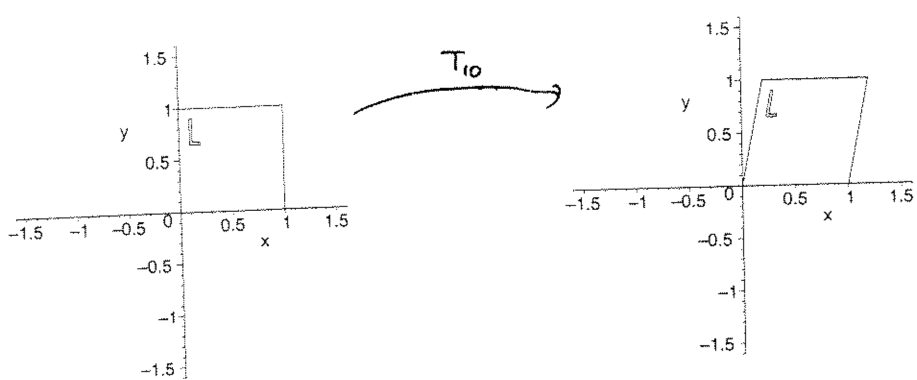
rotate by $\pi/2$ radians counterclockwise



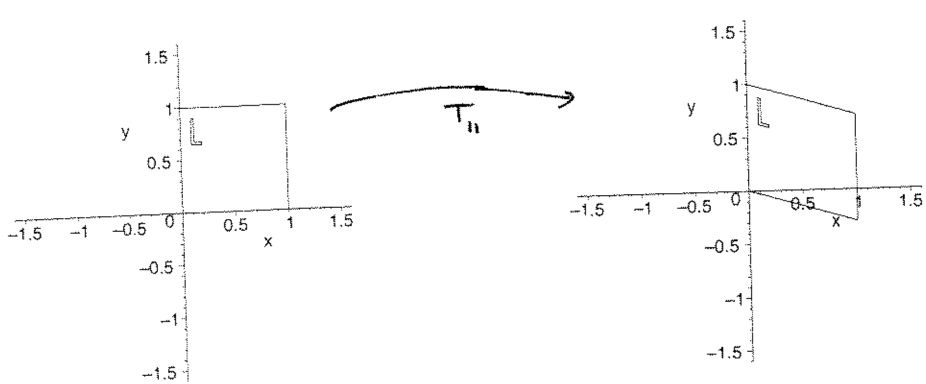
mystery linear trans.



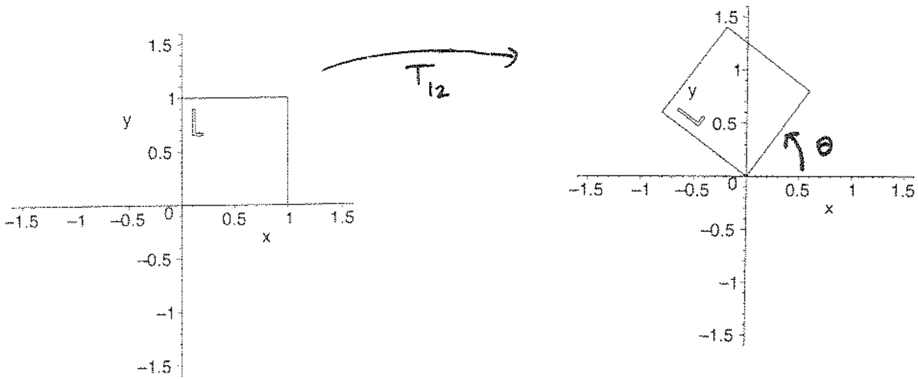
another mystery!



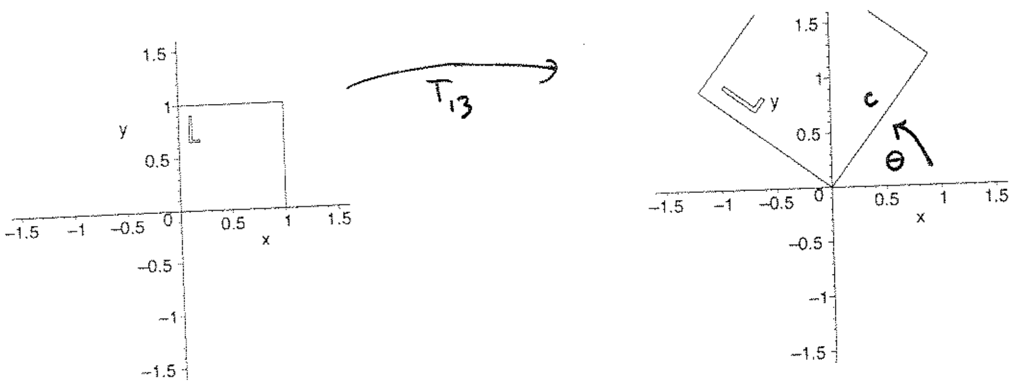
horizontal shear with strength .2



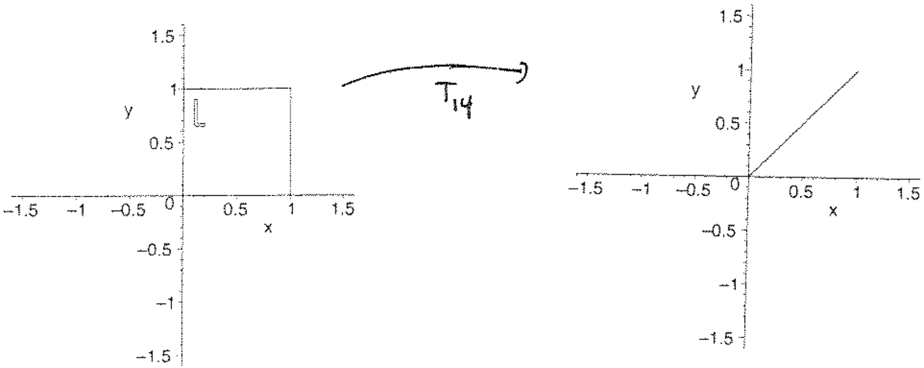
vertical shear with strength -.3



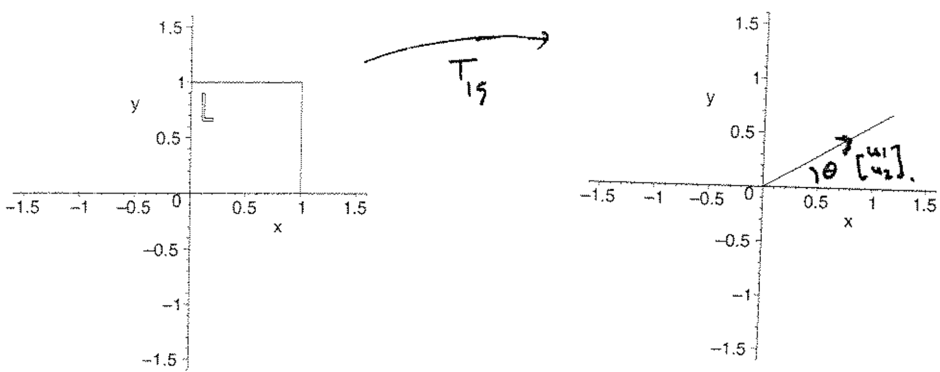
rotation (c.c.) by angle θ



rotate by θ and scale uniformly by factor of c ("rotation dilation")



project onto the line $y=x$



project onto line thru origin at angle θ, with unit direction $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.