

Consider $T(x)=Ax$, for the large matrix

```
> with(LinearAlgebra) :
> A := Matrix(4, 6, [1, 2, 3, -1, -2, 6,
                    0, 1, 1, -2, -1, 3,
                    2, -4, -2, -1, 4, 3,
                    3, -2, 1, -2, 2, 9]);
```

$$A := \begin{bmatrix} 1 & 2 & 3 & -1 & -2 & 6 \\ 0 & 1 & 1 & -2 & -1 & 3 \\ 2 & -4 & -2 & -1 & 4 & 3 \\ 3 & -2 & 1 & -2 & 2 & 9 \end{bmatrix}$$

(1)

```
>
```

1a) Find an explicit representation for $\ker(T)$ (which we also write as $\ker(A)$). Use this representation to find a basis for $\ker(T)$ and verify that it's a basis. You might want to use:

```
> ReducedRowEchelonForm(A);
```

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(2)

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```

1b) As it turns out, Maple has commands to find a basis for the kernel too....the synonym for kernel is nullspace. This word makes clear sense, since "null" means zero, and we are looking for domain vectors that are transformed to zero.

```
> NullSpace(A); #if we're lucky, it's the same as the basis we found – it need not be,
                # since subspace bases are not unique!
```

$$\left\{ \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(3)

2a) For the same transformation, find an explicit representation for $\text{Im}(T)$. Use the fewest number of vectors which still span this subspace. Verify that they are a basis. Hint: reuse your work in the previous part to cull vectors from this "column space." You can read this information from

`> ReducedRowEchelonForm(A);`

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(4)

`>`

Notice column dependencies correspond to solutions to $Ax=0$, which do not change as you do elementary row operations. For example, since the third column of the reduced matrix is the sum of the first two, this must also be true for the original matrix!

2b) A clever way to get an especially nice basis for $\text{Im}(T)$ is to use "elementary column operations" to replace the column vectors (which span $\text{Im}(T)$) with a nicer collection which still spans. This amounts to computing the reduced column echelon form of the matrix:

```
> B := Transpose(A); # turn columns into rows
```

$$B := \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & -4 & -2 \\ 3 & 1 & -2 & 1 \\ -1 & -2 & -1 & -2 \\ -2 & -1 & 4 & 2 \\ 6 & 3 & 3 & 9 \end{bmatrix} \quad (5)$$

```
> C := ReducedRowEchelonForm(B) :
  Transpose(C); #turn rows back into columns; this is the reduced column echelon form!
  #Identify a "nice" basis of Im(T). Check work!
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (6)$$

```
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```

2c) Let's compare our different basis choices for the Image to what Maple comes up with. The image of T is the span of the columns of A , and this subspace is also known by the synonym "column space".

```
> ColumnSpace(A); # is this one of the bases we found? Yipee!
```

$$\left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right] \quad (7)$$

```
>
```

2d) What space do you think Maple is finding a basis for with this command, and is it a subset of the domain or the target?

```
> RowSpace(A);
```

$$\left[\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \right] \quad (8)$$

```
>
```

2e) How is the Row space of A related to the kernel of A ?

