Math 2270-3
Wednesday Sept 16

- Finish pages 3-5 Tuesday notes, then begin §3.2-3.3
- Big matrix example, pages 3-5 today.
- Precise definitions (we've been using these; here's a list).

Def A subset W of $\mathbb{R}^n$ is a subspace iff it is closed under addition and scalar multiplication

i.e. iff whenever $\vec{u}, \vec{v} \in W$ and $k \in \mathbb{R}$, then also

(i) $\vec{u} + \vec{v} \in W$

(ii) $k\vec{u} \in W$

Examples On Tuesday's notes we verified that for $T(\vec{x}) = A\vec{x}$ linear, $T: \mathbb{R}^n \to \mathbb{R}^m$ then

$$\ker(T) = \{\vec{x} \in \mathbb{R}^n \text{ such that } A\vec{x} = 0\}$$

and $$Im(T) = \{y \in \mathbb{R}^m \text{ s.t. there exists } \vec{x} \in \mathbb{R}^n \text{ with } A\vec{x} = \vec{y}\}$$

are subspaces of the domain and target spaces, respectively.

We also showed that the only subspaces of $\mathbb{R}^3$ are

- $\{\vec{0}\}$
- a line thru $\vec{0}$, i.e. $\{t\vec{u} \text{ s.t. } t \in \mathbb{R} \text{ and } \vec{u} \neq \vec{0}\}$
- a plane thru $\vec{0}$, i.e. $\{t\vec{u} + s\vec{v},\ t, s \in \mathbb{R},\ \vec{u}, \vec{v} \text{ non-zero and not multiples}\}$
- all of $\mathbb{R}^3$.

This reasoning generalizes, so that we'll be able to understand all subspaces of $\mathbb{R}^n$.

Def A linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ is any vector $\vec{v}$ which is a sum of multiples of those vectors, i.e.

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k \qquad c_1, c_2, \ldots c_k \in \mathbb{R}.$$

Def the span of $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$:= the collection of all linear combinations, i.e.

$$\text{span}\{\vec{v}_1, \ldots \vec{v}_k\} = \{\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k \text{ s.t. } c_j \in \mathbb{R}\ j = 1, 2, \ldots k\}$$

Examples the span of a single $\vec{v} \neq \vec{0}$ is $\{t\vec{v} \text{ s.t. } t \in \mathbb{R}\}$, i.e. a line thru $\vec{0}$

if $\vec{u}, \vec{v}$ are not scalar multiples, $\vec{u}, \vec{v} \in \mathbb{R}^3$, then their span is a plane thru $\vec{0}$.

<u>Def</u> We shall say that $\vec{v}$ is <u>linearly dependent</u> on $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ (or $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$) if we can write $\vec{v}$ as a linear combination of $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_k$

<u>Example</u> $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ so $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is <u>linearly dependent</u> on $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

<u>Def</u> A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is <u>linearly dependent</u> iff at least one of the vectors $\vec{v}_j$ is lin. dependent on the rest, i.e. a linear combo of the others.

A symmetric way to say $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is linearly dependent is that some linear combination

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0} \qquad \text{where not all } c_j = 0.$$

check!

<u>Def</u> $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is <u>linearly independent</u> iff it is not linearly dependent, i.e. the only way

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0} \quad \text{is if all } c_j = 0$$
$$(c_1 = c_2 = \ldots = c_k = 0)$$

(or equivalently, no $\vec{v}_j$ is a linear combination of the other $\vec{v}_i$'s).

<u>Def</u> A subset $\{\vec{v}_1, \vec{v}_2, \ldots v_k\}$ of a subspace W is a <u>basis</u> for W if $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ <u>spans W</u> and is <u>linearly independent</u>

[this is good!]. In this case we say that the <u>dimension of W</u>, <u>dim(W)</u> = <u>k</u>.

<u>Example</u>: Find a basis for $W = \text{span}\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$. What geometric object is W? What is dim(W)?

<u>Theorem</u> If $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is a basis for W, then each $\vec{w} \in W$ has a <u>unique</u> linear combo representation, $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$.

proof: If also, $\vec{w} = d_1\vec{w}_1 + d_2\vec{w}_2 + \ldots + d_k\vec{v}_k$

then $0 = \vec{w} - \vec{w} = (c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \ldots + (c_k - d_k)\vec{v}_k$

by independence, $c_1 - d_1 = 0, c_2 - d_2 = 0, \ldots c_k - d_k = 0$

so $c_i = d_i$ each i ∎

Math 2270–3
kernel and image – big example
September 16, 2009

Consider T(x)=Ax, for the large matrix

```
> with(LinearAlgebra):
> A := Matrix(4, 6, [1, 2, 3, -1, -2, 6,
                     0, 1, 1, -2, -1, 3,
                     2, -4, -2, -1, 4, 3,
                     3, -2, 1, -2, 2, 9]);
```

$$A := \begin{bmatrix} 1 & 2 & 3 & -1 & -2 & 6 \\ 0 & 1 & 1 & -2 & -1 & 3 \\ 2 & -4 & -2 & -1 & 4 & 3 \\ 3 & -2 & 1 & -2 & 2 & 9 \end{bmatrix} \quad (1)$$

```
>
```

1) Find an explicit representation for ker(T) (which we also write as ker(A)). Use this representation to find a basis for ker(T) and verify that it's a basis. You might want to use:

```
> ReducedRowEchelonForm(A);
```

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2)$$

```
>
```

2a) For the same transformation, find an explicit representation for Im(T). Use the fewest number of vectors which still span this subspace. Verify that they are a basis. Hint: reuse your work in the previous part to cull vectors from this "column space." You can read this information from

> *ReducedRowEchelonForm*(*A*);

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3)$$

>

Notice column dependencies correspond to solutions to Ax=0, which do not change as you do elementary row operations. For example, since the third column of the reduced matrix is the sum of the first two, this must also be true for the original matrix!

2b) A clever way to get an especially nice basis for Im(T) is to use "elementary column operations" to replace the column vectors (which span Im(T)) with a nicer collection which still spans. This amounts to computing the reduced column echelon form of the matrix:

```
> B := Transpose(A); #turn columns into rows
```

$$B := \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & -4 & -2 \\ 3 & 1 & -2 & 1 \\ -1 & -2 & -1 & -2 \\ -2 & -1 & 4 & 2 \\ 6 & 3 & 3 & 9 \end{bmatrix} \qquad (4)$$

```
> C := ReducedRowEchelonForm(B) :
  Transpose(C); #turn rows back into columns; this is the reduced column echelon form!
                #Identify a "nice" basis of Im(T). Check work!
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad (5)$$

```
>
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