Math 2270-3
Tuesday Sept 15
§3.1 Image and kernel of linear transformations

HW for Friday Sept 18 (was posted last Fri.)
2.3 29, 33, 34, 38, 45, 48, 49, 50
2.4 19, 20, 21, 23, 25, 29, 33, 38, 67, 68, 69, 70, 71, 73, 104 ↑ justify!
Review p 98-99: justify! 3, 5, 6, 7, 8, 10, 16, 26, 34, 40, 41
3.1 3, 5, 6, 13, 19, 21, 23, 24, 25, 32, 33, 34, 37.

Recall,
If $f: X \to Y$ is a function

X is called <u>domain</u> of f
Y is called <u>target</u> of f ← to avoid confusion because the usual word "range" is sometimes ambiguous.
$\{ f(x) \in Y \text{ s.t. } x \in X \}$ is a subset of Y and is called <u>image</u> of f

<u>Ex</u> $f: \mathbb{R} \to \mathbb{R}$
$f(x) = e^x$
domain $= \mathbb{R}$
target $= \mathbb{R}$
image $= \{ y \in \mathbb{R} \text{ s.t. } y > 0 \} = (0, \infty)$

<u>Today</u>: $T: \mathbb{R}^n \to \mathbb{R}^m$ linear, i.e. $\begin{cases} T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \\ T(k\vec{u}) = kT(\vec{u}) \end{cases} \quad \forall \vec{u}, \vec{v} \in \mathbb{R}^n, k \in \mathbb{R}$

- then what are the possible geometries of image(T)?
- also, the <u>kernel of T</u>, ker(T) is defined to be the domain subset mapped to $\vec{0}$:

$$\ker(T) := \{ \vec{x} \in \mathbb{R}^n \text{ s.t. } T(\vec{x}) = \vec{0} \}.$$

= "vectors that get squashed by T"

What ~~is~~ are the possible geometries of ker(T)?

Examples for $T: \mathbb{R}^2 \to \mathbb{R}^2$ linear:

	kernel	image
rotations		
reflections across a line L thru $\vec{0}$		
projections onto L		
invertible T		

<u>Theorem</u> (3.1.8) The following are <u>equivalent</u> characterizations that the linear transformation $T(\vec{x}) = A\vec{x}$, $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible

(1) (rref)

(2) (ker)

(3) (image)

(4) (1-1)

(5) (onto)

(6) (rank)

(7) (a unique sol.).

example $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -4 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$$

ker(f) is easy

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 2 & -4 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -R_1+R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ \tfrac{1}{2}R_2 \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} R_2+R_1 \\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$
$x_2 = 2t$
$x_1 = -t$

$$\boxed{\ker f = \left\{ t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}}$$

line thru origin.

image(f) a little harder

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & y_1 \\ 0 & 2 & -4 & y_2 \\ 1 & 1 & -1 & y_3 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -R_1+R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 3 & y_1 \\ 0 & 2 & -4 & y_2 \\ 0 & 2 & -4 & y_3-y_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & y_1 \\ 0 & 1 & -2 & y_2/2 \\ 0 & 0 & 0 & y_3-y_1-y_2 \end{array}\right]$$

this will be solvable iff $-y_1 - y_2 + y_3 = 0$

i.e. iff $\vec{y}$ lies on the plane thru origin $\left\{\vec{y} \in \mathbb{R}^3 : y_1 + y_2 - y_3 = 0\right\}$

or (better!)

$$\text{Image}(f) = \left\{ x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

$$= \left\{ \text{all linear combos of } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} \right\}.$$

$$:= \underline{\text{span}}\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$

What can I do to a collection of vectors to clean them up and figure out their span?

1. interchange 2 vectors
2. multiply a vector by non-zero const
3. replace a vector by its sum with a scalar multiple of another vector

the ker(f) tells me about possible redundancy in my "span" expression for image(f)

expressing $f(\vec{x})$ in the linear combo of cols way, and looking at ker(f), we see

$$-1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} = 0$$

so $\begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. so $\text{span}\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\} = \text{span}\left\{\vec{v}_1, \vec{v}_2\right\}$ is a plane!

$\vec{v}_3 = \vec{v}_2 - 2\vec{v}_1$

What kind of subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ can $\ker T$ and image(T) be, when T is linear?

Theorem ker( ) is closed under addition and scalar multiplication

proof: We need to show that if $\vec{u}, \vec{v} \in \ker T$ and $k \in \mathbb{R}$, then

(1) $\vec{u} + \vec{v} \in \ker T$

(2) $k\vec{u} \in \ker T$

check:

Theorem image( ) is closed under addition and scalar multiplication.

proof: We need to show that if $\vec{w}, \vec{z} \in$ image(T) & $k \in \mathbb{R}$ then

(1) $\vec{w} + \vec{z} \in$ image(T)

(2) $k\vec{w} \in$ image(T)

check:

So, what subsets of $\mathbb{R}^n$ are closed under addition and scalar multiplication?
e.g. in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$?

possible subsets of $\mathbb{R}^3$ closed under addition and scalar multiplication

Let S be such a subset (called a <u>subspace</u>)

↑ a subspace of $\mathbb{R}^n$ is a subset closed under addition and scalar multiplication. This page shows the only <u>subspaces</u> of $\mathbb{R}^3$ are

(0) $\{\vec{0}\}$
(1) Lines thru origin
(2) planes thru origin
(3) all of $\mathbb{R}^3$

Let $\vec{u} \in S$

then $0\vec{u} = \vec{0}$ is a scalar mult, so

$$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\} \in S$$

↑

0 One possibility [closed under + & s.m]

If this is not all of S, then let $\vec{u} \neq \vec{0}$, $\vec{u} \in S$.

$\Rightarrow \{t\vec{u}\} \subset S$ ← 1 another possibility, a line thru origin

If $L = \{t\vec{u} : t \in \mathbb{R}\}$ is not all of S, Let $\vec{v} \in S$, $\vec{v} \notin L$

$$\Rightarrow \{t\vec{u} + s\vec{v} : s, t \in \mathbb{R}\} \subset S$$

↑ 2 plane thru origin.

[could get normal vector by $\vec{u} \times \vec{v}$ or rref $\left[\begin{array}{c|c} \vec{u} & 0 \\ \vec{v} & 0 \end{array}\right]$]

If the plane above is not all of S,
Let $\vec{w} \in S$, $\vec{w} \notin$ plane.

then $\{t\vec{u} + s\vec{v} + r\vec{w} : s, t, r \in \mathbb{R}\} = \mathbb{R}^3$! 3

<u>proof</u>: this is "clear" geometrically but requires an algebraic proof. The explanation is quite clever, with ideas <u>we shall use over & over</u>:

Write $t\vec{u} + s\vec{v} + r\vec{w} = \underbrace{\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}}_{A} \begin{bmatrix} t \\ s \\ r \end{bmatrix}$

from page 2,
the transformation $T(\vec{x}) = A\vec{x}$ is onto iff $\ker(T) = \{\vec{0}\}$ ← i.e. S is all of $\mathbb{R}^3$.

Suppose $A\begin{bmatrix} t \\ s \\ r \end{bmatrix} = \vec{0}$. Then $t\vec{u} + s\vec{v} + r\vec{w} = \vec{0}$

if $r \neq 0$, then solve for $\vec{w}$: $\vec{w} = -\frac{t}{r}\vec{u} - \frac{s}{r}\vec{v}$
but we chose $\vec{w}$ <u>not</u> in $\vec{u}$-$\vec{v}$ plane!

thus $r = 0$, so $t\vec{u} + s\vec{v} = \vec{0}$
thus $s = 0$, since otherwise $\vec{v} = -\frac{t}{s}\vec{u}$.
thus $t = 0$, since otherwise $\vec{u} = \vec{0}$!

so $\ker T = \{0\}$

so our space is all of $\mathbb{R}^3$!