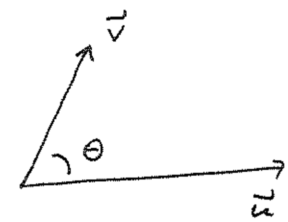
Math 2270-3
Tuesday Sept. 1.

Appendix A & 2.1

Recall the dot product $\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i v_i$, for $\vec{u},\vec{v}\in\mathbb{R}^n$.

On Monday we finished proving (for $n=2,3$)

<u>Theorem</u> $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$



<u>Definition</u> We say $\vec{u},\vec{v}$ are <u>perpendicular</u> or <u>orthogonal</u> iff $\vec{u}\cdot\vec{v}=0$ ($\vec{u},\vec{v}\in\mathbb{R}^n$!)
(i.e., by Thm above, $\cos\theta=0$ so $\theta=\pi/2$, at least in $\mathbb{R}^2,\mathbb{R}^3$.)
We write $\vec{u}\perp\vec{v}$.

<u>Example</u>: Are $\begin{bmatrix}1\\-1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\0\\-1\\0\end{bmatrix}$ orthogonal?

<u>Example</u>: Find an implicit equation for all points (x,y,z) whose position vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$ is perpendicular to $\begin{bmatrix}-1\\2\\-2\end{bmatrix}$. What geometric object is described by these points?

<u>Example</u>: Find an implicit equation for all points (x,y,z) so that the displacement vector from $(2,1,0)$ to (x,y,z) is perpendicular to $\begin{bmatrix}-1\\2\\-2\end{bmatrix}$. What object is this?

<u>Exercise</u>: Explain why the solution set to $ax+by+cz=d$ is a plane in $\mathbb{R}^3$.
Hint: Let (x_0, y_0, z_0) satisfy the equation. Then rewrite the equation using dot product, as in previous example.

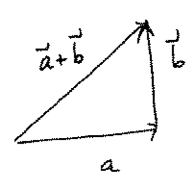
<u>Example</u> Find all vectors in $\mathbb{R}^3$ perpendicular to both $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(You could do this using cross product, but there's a general "LS" way, which works for any number of vectors in $\mathbb{R}^n$.)

<u>Easy Pythagorean Theorem in $\mathbb{R}^n$</u>: Let $\vec{a}, \vec{b} \in \mathbb{R}^n$, $\vec{a} \perp \vec{b}$ (i.e. $\vec{a} \cdot \vec{b} = 0$)

Then $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$

(and, this formula holds if and only if $\vec{a} \perp \vec{b}$.)



<u>proof</u>: $\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b})$
$= \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\,\vec{a} \cdot \vec{b}$. ∎

We'll use dot product alot is Chapters 2 & 5.

§ 2.1 Matrix (Linear) Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

$\vec{T}(\vec{x}) = A\vec{x}$

$\vec{x} \in \mathbb{R}^n$, $A_{m\times n}$, $\vec{T}(\vec{x}) \in \mathbb{R}^m$

↑

matrix transformation

(also, $\vec{T}(\vec{x}) = A\vec{x} + \vec{b}$ "affine transformation")

Remember from last week that for

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \text{Row}_1(A) \\ \text{row}_2(A) \\ \vdots \\ \text{row}_m(A) \end{bmatrix} = \left[\text{col}_1(A) \,\vdots\, \text{col}_2(A) \,\vdots\, \cdots \text{col}_n(A) \right]$$

$$\text{entry}_{ij}(A) = a_{ij} = \text{entry}_j(\text{row}_i(A)) = \text{entry}_i(\text{col}_j(A))$$

and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Then
$$A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \underbrace{\begin{bmatrix} \text{row}_1(A)\cdot\vec{x} \\ \text{row}_2(A)\cdot\vec{x} \\ \vdots \\ \text{row}_m(A)\cdot\vec{x} \end{bmatrix}}_{\text{dot product form}} = \underbrace{x_1\,\text{col}_1(A) + x_2\,\text{col}_2(A) + \dots + x_n\,\text{col}_n(A)}_{\text{linear combination form.}}$$

Aside:

Matrix transformations arise in myriad applications.
One place you've seen them (probably without realizing it) is differential (or tangent) approximations in calculus.

<u>1-variable</u>: $\underbrace{\underbrace{g(x_0+h) - g(x_0) = g'(x_0)h}_{dg} + h\varepsilon(h)}_{\Delta g}$ $\qquad \varepsilon(h) \to 0$ as $h \to 0$

$dg: h \mapsto \underbrace{g'(x_0)}_{1\times 1 \text{ matrix}} h$

$g: \mathbb{R}^n \to \mathbb{R}$ $\qquad g(\vec{x}_0 + \vec{h}) - g(\vec{x}_0) = \underbrace{\left[\frac{\partial g}{\partial x_1}(\vec{x}_0), \frac{\partial g}{\partial x_2}(\vec{x}_0) \cdots \frac{\partial g}{\partial x_n}(\vec{x}_0)\right]}_{\text{gradient}} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} + \vec{h}\cdot\vec{\varepsilon}(\vec{h})$ $\qquad dg: \vec{h} \mapsto \underbrace{\nabla g(\vec{x}_0)}_{1\times n \text{ matrix.}} \cdot \vec{h}$

$\vec{g}: \mathbb{R}^n \to \mathbb{R}^m$?

Geometry and algebra of linear transformations

Theorem: Let $T(\vec{x}) = A\vec{x}$ be a matrix transformation, $T: \mathbb{R}^n \to \mathbb{R}^m$, $A_{m\times n}$.
Then

(a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \forall \vec{u},\vec{v} \in \mathbb{R}^n$

(b) $T(t\vec{u}) = t\,T(\vec{u}) \quad \forall\, t\in\mathbb{R},\ \vec{u}\in\mathbb{R}^n$

Check: $\text{entry}_i\left(A(\vec{u}+\vec{v})\right) = \text{row}_i(A)\cdot(\vec{u}+\vec{v}) = \text{row}_i(A)\cdot\vec{u} + \text{row}_i(A)\cdot\vec{v}$
$= \text{entry}_i(A\vec{u}) + \text{entry}_i(A\vec{v})$ ■

$\text{entry}_i\left(A(t\vec{u})\right) = \text{row}_i(A)\cdot(t\vec{u}) = t\,\text{row}_i(A)\cdot\vec{u}$
$= t\ \text{entry}_i(A\vec{u})$ ■

(c) Let L be a line in the domain $\mathbb{R}^n$.
Let L be given explicitly as points with position vectors $\{\vec{a} + t\vec{b},\ t\in\mathbb{R}\}$
$\uparrow$ $\vec{b}\neq\vec{0}$ direction vector.

Then L is transformed by T into a line in $\mathbb{R}^m$,
if $T(\vec{b})\neq\vec{0}$.
If $T(\vec{b})=\vec{0}$, then L is squashed into a point.

check: The image of L is $\{T(\vec{a}+t\vec{b}),\ t\in\mathbb{R}\} := T(L)$
$= \{T(\vec{a}) + t\,T(\vec{b}),\ t\in\mathbb{R}\}$
is a line thru $T(a)$
with dir $T(\vec{b})$ iff $T(b)\neq\vec{0}$.
Else is always $T(\vec{a})$ ■

(d) If L is transformed to a line by T,
then every line parallel to L is transformed to a line parallel to $T(L)$.

check parallel lines can be represented explicitly with identical direction vectors, so if L is ~~abc~~ as above, and L' is parallel, then $L' = \{\vec{a}' + t\vec{b},\ t\in\mathbb{R}\}$

so $T(L') = \{T(\vec{a}') + t\,T(\vec{b}),\ t\in\mathbb{R}\}$
$\uparrow$
same dir vect as L

■

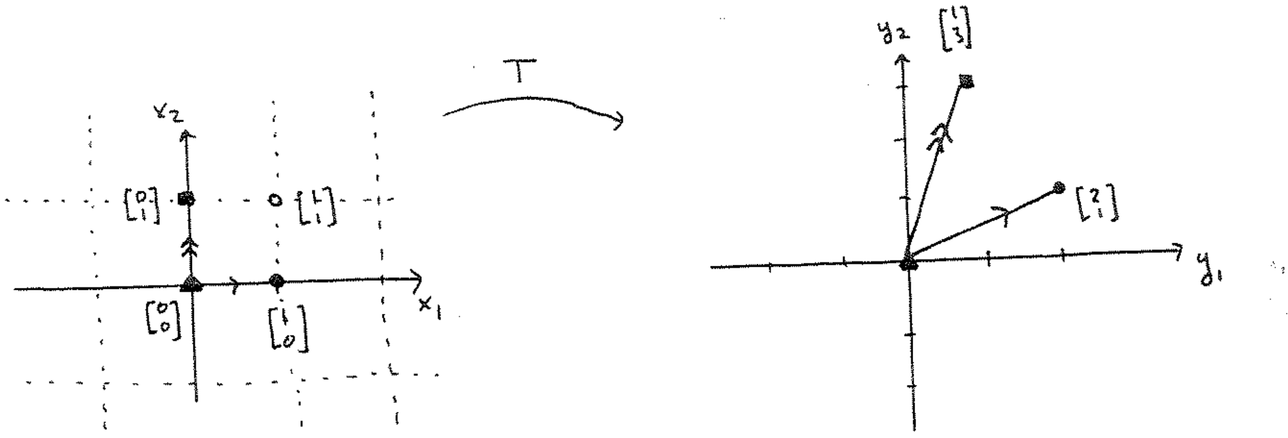
<u>Example</u> Consider the transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$= \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

(See also the nice story problem page 40-41.)

Complete the following diagram which shows how T transforms the plane. Use facts from the previous page:



Does T have an inverse transformation?

... If $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is given, to solve $T(\vec{x}) = \vec{y}$

rref $A \vdots \vec{y}$, i.e.

$$\left[\begin{array}{cc|c} 2 & 1 & y_1 \\ 1 & 3 & y_2 \end{array}\right]$$

to be cont'd!