

Math 2270-3

Friday Oct 9

Finish §4.2, begin §4.3

(some more §4.3 after break)

- rank + nullity theorem, page 3 Wed.
- isomorphism discussion, page 4 Wed.

point (e) details, expanded:

$T: V \rightarrow W$ linear.

T is an isomorphism iff $T^{-1}: W \rightarrow V$ exists.

- Thus T isomorphism $\Rightarrow T$ is 1-1. Since $T(0) = 0$, $\ker T = \{0\}$.
 $\Rightarrow T$ is onto. Thus $\text{image}(T) = W$.

- T linear, and rank + nullity theorem
 \Rightarrow that if $\ker T = \{0\}$, then $\text{Image}(T) = W$
(since image is n -dim'l subspace of W it is all of W)

also \Rightarrow that if $\text{Image}(T) = W$ then $\ker(T) = \{0\}$

- If $\ker T = \{0\}$ then T is 1-1, since $T(f) = T(g) \Rightarrow T(f-g) = 0 \Rightarrow f-g = 0$ ($\ker T = \{0\}$)

Thus, $\left. \begin{matrix} \ker T = \{0\} \\ \text{Image}(T) = W \end{matrix} \right\} \Rightarrow T \text{ is 1-1 \& onto.} \Rightarrow T^{-1} \text{ exists} \Rightarrow T \text{ is isomorphism.}$

We proved, for $T: V \rightarrow W$ linear, $\dim V = \dim W$, that:

$$T \text{ is an isomorphism} \begin{matrix} \Rightarrow \\ \Leftarrow \end{matrix} \left\{ \begin{matrix} \ker T = \{0\} \\ \Updownarrow \\ \text{Image } T = W \end{matrix} \right. \text{ which is (e).}$$

§ 4.3

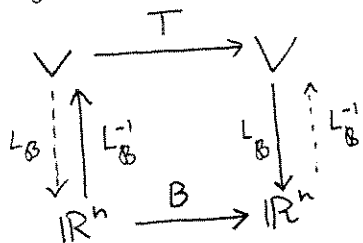
Matrix of a linear transformation with respect to a basis

Let $T: V \rightarrow V$ linear.

$B = \{f_1, f_2, \dots, f_n\}$ a basis for V . Let $L_B: V \rightarrow \mathbb{R}^n$ be the coordinate transformation: $L_B(f) = [f]_B$.

Then $B := [T]_B :=$ the matrix for T with respect to B is

defined by this diagram:



column by column:
 $f_j \in V \rightarrow T(f_j) \in V$
 $\uparrow \qquad \qquad \downarrow$
 $\vec{e}_j \in \mathbb{R}^n \dashrightarrow [T(f_j)]_B$
 so $\text{col}_j(B) = [T(f_j)]_B$

By definition,

$$[T(c_1 f_1 + c_2 f_2 + \dots + c_n f_n)]_B = L_B \circ T(c_1 f_1 + c_2 f_2 + \dots + c_n f_n) = L_B \circ T \circ L_B^{-1} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$\vec{c} \mapsto L_B \circ T \circ L_B^{-1}(\vec{c})$
 is composition of linear, so is linear,
 domain = \mathbb{R}^n
 target = \mathbb{R}^n
 so it is given by matrix multiplication for a matrix B ;
 $\vec{c} \mapsto B\vec{c}$
 $B := [T]_B$.

Examples: For $T(x) = Ax$, $A_{n \times n}$
 $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ an alternate basis for \mathbb{R}^n , this is exactly what we did in chapter 3.

Example 1: Consider the derivative operator $D: \mathbb{P}_2 \rightarrow \mathbb{P}_2$
 $a_0 + a_1 x + a_2 x^2 \in \mathbb{P}_2 \xrightarrow{D} a_1 + 2a_2 x \in \mathbb{P}_2$
 Fill in the diagram at the right to find the matrix for D with respect to the basis $B = \{1, x, x^2\}$
 $\in \mathbb{R}^3 \xrightarrow{\quad} \in \mathbb{R}^3$
 (Diagram with L_B^{-1} and L_B arrows)

Check: column by column
 ans: $[D]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Example 2 Using the coordinate isomorphism allows you to systematically solve kernel and image problems (actually all problems), for linear transformations $T: V \rightarrow V$, when $\dim(V) < \infty$.

Our running example (4) has been

$$T: M_{2 \times 2} \rightarrow M_{2 \times 2}$$
$$T(M) := M \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} M$$

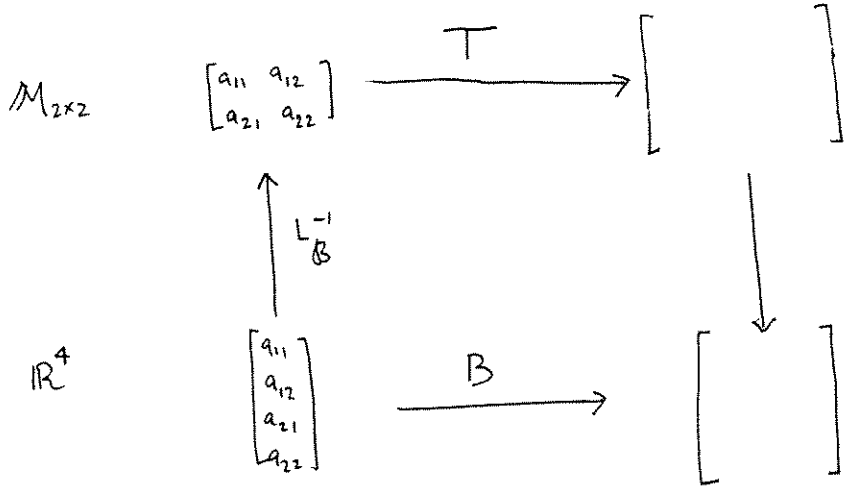
previously:

$$\ker(T) = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$
$$\text{image}(T) = \text{span} \left\{ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Fill in the commutative diagram below, to find $B = [T]_{\mathcal{B}}$,

for $\mathcal{B} = \{ E_{11}, E_{12}, E_{21}, E_{22} \}$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix}$$



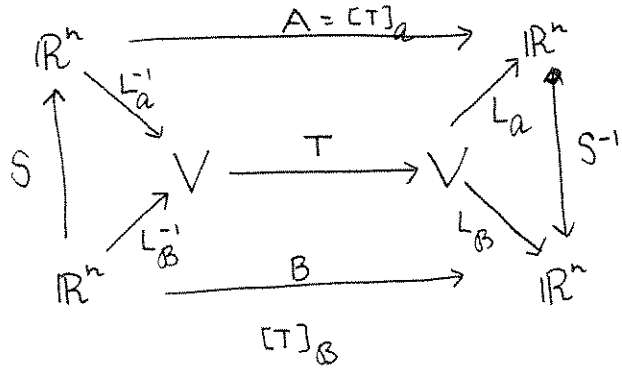
Use B; find $\ker B$. } chptr 3
image B. }

Deduce $\ker(T)$ } chptr 4!
image(T)!

Let $T: V \rightarrow V$ linear. $\dim V = n < \infty$.

How does the matrix for T change, if you change bases?

Here's how! $\mathcal{B} = \{f_1, f_2, \dots, f_n\}$; $\mathcal{A} = \{g_1, g_2, \dots, g_n\}$.



$$B = S^{-1} A S_{\mathcal{A} \leftarrow \mathcal{B}} !$$