Math 2270-3
Friday Oct 9
Finish §4.2, begin §4.3
(some more §4.3 after break)

- rank + nullity theorem, page 3 Wed.
- isomorphism discussion, page 4 Wed.

point (e) details, expanded:

$T: V \to W$ linear.

T is an isomorphism iff $T^{-1}: W \to V$ exists.

- Thus T isomorphism $\Rightarrow T$ is 1-1. Since $T(0)=0$, $\ker T = \{0\}$.
  $\Rightarrow T$ is onto. Thus image$(T) = W$.

- T linear, and rank + nullity theorem
  $\Rightarrow$ that if $\ker T = \{0\}$, then Image$(T) = W$
  (since image is n-dim'l subspace of W it is all of W)

  also $\Rightarrow$ that if Image$(T) = W$ then $\ker(T) = \{0\}$

- If $\ker T = \{0\}$ then T is 1-1, since $T(f)=T(g) \Rightarrow T(f-g)=0$
  $\Rightarrow f-g=0$ ($\ker T = \{0\}$)

  Thus, $\left.\begin{array}{l}\ker T = \{0\} \\ \text{Image}(T) = W\end{array}\right\} \Rightarrow T$ is 1-1 & onto.
  $\Rightarrow T^{-1}$ exists $\Rightarrow T$ is isomorphism.

We proved, for $T: V \to W$ linear, $\dim V = \dim W$, that:

$$T \text{ is an isomorphism} \;\substack{\Rightarrow \\ \Leftarrow}\; \left\{\begin{array}{l}\ker T = \{0\} \\ \quad\Updownarrow \\ \text{Image } T = W\end{array}\right.$$

which is (e).

§4.3
Matrix of a linear transformation with respect to a basis

Let $T: V \to V$ linear.
$\mathcal{B} = \{f_1, f_2, \ldots f_n\}$ a basis for V. Let $L_{\mathcal{B}}: V \to \mathbb{R}^n$ be the coordinate transformati[on] $L_{\mathcal{B}}(f) = [f]_{\mathcal{B}}$.

Then $B := [T]_{\mathcal{B}}$:= the matrix for T with respect to $\mathcal{B}$ is defined by this diagram:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, dashed, shift right, "L_{\mathcal{B}}"'] & V \arrow[d, shift right, "L_{\mathcal{B}}"'] \\ \mathbb{R}^n \arrow[r, "B"] \arrow[u, shift right, "L_{\mathcal{B}}^{-1}"'] & \mathbb{R}^n \arrow[u, dashed, shift right, "L_{\mathcal{B}}^{-1}"'] \end{tikzcd}$$

column by column:

$$\begin{tikzcd} f_j \in V \arrow[r] & T(f_j) \in V \arrow[d] \\ \vec{e}_j \in \mathbb{R}^n \arrow[u] \arrow[r, dashed] & {[T(f_j)]_{\mathcal{B}}} \end{tikzcd}$$

so $\mathrm{col}_j(B) = [T(f_j)]_{\mathcal{B}}$

By definition,

$$[T(c_1f_1 + c_2f_2 + \cdots + c_nf_n)]_{\mathcal{B}}$$
$$= L_{\mathcal{B}} \circ T\,(c_1f_1 + c_2f_2 + \cdots + c_nf_n)$$
$$= L_{\mathcal{B}} \circ T \circ L_{\mathcal{B}}^{-1} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$\vec{c} \mapsto L_{\mathcal{B}} \circ T \circ L_{\mathcal{B}}^{-1}(\vec{c})$

is composition of linear, so is linear,
domain $= \mathbb{R}^n$
target $= \mathbb{R}^n$
so it is given by matrix multiplication for a matrix B;

$\vec{c} \to B\vec{c}$

$B := [T]_{\mathcal{B}}$.

<u>Examples</u>: For $T(\vec{x}) = A\vec{x}$, $A_{n\times n}$
$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ an alternate basis for $\mathbb{R}^n$, this is exactly what we did in chapter 3.

<u>Example 1</u>: Consider the derivative operator $D: \mathbb{P}_2 \to \mathbb{P}_2$

$$\begin{tikzcd} a_0 + a_1x + a_2x^2 \in \mathbb{P}_2 \arrow[r, mapsto, "D"] & a_1 + 2a_2x. \in \mathbb{P}_2 \arrow[d, "L_{\mathcal{B}}"] \\ \in \mathbb{R}^3 \arrow[u, "L_{\mathcal{B}}^{-1}"] \arrow[r, mapsto] & {} \end{tikzcd}$$

Fill in the diagram at the right to find the matrix for D with respect to the basis
$\mathcal{B} = \{1, x, x^2\}$
Check: column by column

ans: $[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

<u>Example 2</u> Using the coordinate isomorphism allows you to systematically solve kernel and image problems (actually all problems), for linear transformations $T: V \to V$, when $\dim(V) < \infty$.

Our running example (4) has been

$$T: \mathbb{M}_{2\times 2} \to \mathbb{M}_{2\times 2}$$

$$T(M) := M\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} M$$

previously:

$\ker(T) = \operatorname{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

$\operatorname{image}(T) = \operatorname{span}\left\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

Fill in the commutative diagram below, to find $B = [T]_{\mathcal{B}}$,

for $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{tikzcd}
\mathbb{M}_{2\times 2} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \arrow[r, "T"] & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \arrow[d] \\
\mathbb{R}^4 & \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} \arrow[u, "L_{\mathcal{B}}^{-1}"] \arrow[r, "B"] & \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}
\end{tikzcd}$$

Use B; find $\ker B$. } Chptr 3
image B.

Deduce $\ker(T)$ } Chptr 4!
image(T)!

Let $T: V \to V$ linear. $\dim V = n < \infty$.

How does the matrix for T change, if you change bases?

Here's how! $\mathcal{B} = \{f_1, f_2, \dots f_n\}$; $\mathcal{A} = \{g_1, g_2, \dots g_n\}$.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, "A = {[T]_{\mathcal{A}}}"] \arrow[dr, "L_{\mathcal{A}}^{-1}"] & & & \mathbb{R}^n \arrow[dd, "S^{-1}"] \\
& V \arrow[r, "T"] & V \arrow[ur, "L_{\mathcal{A}}"] \arrow[dr, "L_{\mathcal{B}}"'] & \\
\mathbb{R}^n \arrow[uu, "S"] \arrow[ur, "L_{\mathcal{B}}^{-1}"'] \arrow[rrr, "B", "{[T]_{\mathcal{B}}}"'] & & & \mathbb{R}^n
\end{tikzcd}$$

$B = S^{-1} A \, S_{\mathcal{A} \leftarrow \mathcal{B}}$!