Math 2270-3
Wednesday Oct 7

§4.1, 4.2 HW due Friday.
§4.3 HW postponed until Friday after break
- we cover 4.3 this Friday; if at all possible don't miss class!

§4.2 Linear transformations and isomorphisms.
(Generalizes chapter 3)

<u>Def</u>: Let $T: V \to W$ be a function with domain V and target W, where both of V & W are linear spaces.

Then T is called a <u>linear transformation</u> (or "linear") iff

$$\text{(i)} \quad T(f+g) = T(f) + T(g) \qquad \forall f, g \in V$$
$$\text{(ii)} \quad T(kf) = k\,T(f) \qquad \forall f \in V, k \in \mathbb{R}.$$

<u>Examples</u>: We proved in chapter 3 that the linear transformations

1) $T: \mathbb{R}^n \to \mathbb{R}^m$ are precisely the matrix transformations $T(\vec{x}) := A\vec{x}$.

2) Coordinates: If $\mathcal{B} = \{f_1, f_2 .. f_n\}$ is a basis for V then $T(f) := [f]_{\mathcal{B}}$ is a linear transformation, $T: V \to \mathbb{R}^n$.

Also $T^{-1}: \mathbb{R}^n \to V$ is linear, $T^{-1}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$.

(we proved this yesterday; $[f+g]_{\mathcal{B}} = [f]_{\mathcal{B}} + [g]_{\mathcal{B}}$
$[kf]_{\mathcal{B}} = k[f]_{\mathcal{B}}$)

3) Derivative operators, e.g.

$D: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$

$D(f) := f'$ is linear transformation, since $(f+g)' = f' + g'$
$(kf)' = kf'$.

4) (From example 3 yesterday),

$T: M_{2\times2} \to M_{2\times2}$ $\quad T(A) := A\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A$

<u>check this T is linear</u>

<u>Def</u>. Let $T: V \to W$ be linear. Then

$\ker T := \{ f \in V \text{ s.t. } T(f) = 0 \}$
$\text{image}(T) := \{ g \in W \text{ s.t. } g = T(f) \text{ for some } f \in V \}.$

<u>Theorem</u> $\ker(T) \subset V$ and $\text{image}(T) \subset W$ are <u>subspaces</u>
check this! (great test question).

<u>kerT is a subspace</u>:

<u>image(T) is a subspace</u>:

<u>Examples</u>: discuss kernels & images:

1) $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x}) = A\vec{x}$.

2) $T(f): [f]_{\mathcal{B}}$ where $\mathcal{B} = \{f_1, f_2 .. f_n\}$ is a basis for V
$T: V \to \mathbb{R}^n$

3) $D: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$
$D(f) := f'$

4) $T: M_{2\times2} \to M_{2\times2}$ $\quad T(A) := A \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A$

hint: yesterday we worked out $\ker T = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$.
today work out image(T).

<u>Theorem</u>: Let $T: V \to W$ be linear. Let $\dim(V) < \infty$.
Then

$$\underbrace{\dim(\ker T)}_{\text{"nullity of } T\text{"}} + \underbrace{\dim(\mathrm{image}(T))}_{\text{"rank of } T\text{"}} = \dim(V)$$

this is the rank + nullity theorem.

<u>exples</u>: check the 4 running examples

<u>proof</u> (slightly new!). Let $\dim(V) = n$.
Let $\{f_1, f_2 .. f_k\}$ be a basis for $\ker T$, $0 \le k \le n$.
($k=0$ means $\ker T = \{0\}$).

By successively adding vectors, complete this to a basis for all of V:

$$\mathcal{B} = \{f_1, f_2, \dots f_k, f_{k+1}, \cdots f_n\}$$

We claim $\{T(f_{k+1}), T(f_{k+2}), \dots T(f_n)\}$ is a basis for image(T).
(If our claim is correct, it proves the theorem!)

<u>span</u>: $T(c_1 f_1 + c_2 f_2 + \dots + c_k f_k + c_{k+1} f_{k+1} + \dots + c_n f_n)$

$$= c_1 \cdot 0 + c_2 \cdot 0 + \dots + c_k \cdot 0 + c_{k+1} T(f_{k+1}) + c_{k+2} T(f_{k+2}) + \dots + c_n T(f_n)$$

so $\{T(f_{k+1}), T(f_{k+2}), \dots T(f_n)\}$ <u>spans</u> image(T)

<u>independence</u>: If $c_{k+1} T(f_{k+1}) + \dots + c_n T(f_n) = 0$

then $T(\underbrace{c_{k+1} f_{k+1} + \dots + c_n f_n}_{\in \ker(T)!}) = 0$,

$$\Rightarrow c_{k+1} f_{k+1} + \dots + c_n f_n = \underbrace{a_1 f_1 + a_2 f_2 + \dots + a_k f_k}_{\text{basis for } \ker T}$$

independence of $\{f_1, \dots f_k, f_{k+1}, \dots f_n\}$

$$\Rightarrow c_{k+1} = c_{k+2} = \dots = c_n = 0 \quad \blacksquare$$

Def: Let $T: V \to W$ be a linear transformation.

If T has an inverse transformation $T^{-1}: W \to V$ then T is called an isomorphism (equal shape), and V and W are called isomorphic

Examples 1) $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x}) = A\vec{x}$ is an isomorphism iff $m=n$ and A^{-1} exists (iff $\mathrm{rref}(A) = I$, etc.)

2) If $\mathcal{B} = \{f_1, f_2, \ldots f_n\}$ is a basis for V, then $T(f) = [f]_{\mathcal{B}}$, $T: V \to \mathbb{R}^n$ is an isomorphism.

Theorem a) Compositions of linear transformations are also linear transformations

b) If $T: V \to W$ is linear, and has an inverse function $T^{-1}: W \to V$, then T^{-1} is linear.

c) If V and W have the same finite dimension n, then V and W are isomorphic

d) If $n = \dim V \neq \dim W = m$, then V and W are not isomorphic

e) If $n = \dim V = \dim W$ and $T: V \to W$ is linear, then T is an isomorphism iff $\ker T = \{0\}$ (nullity $=0$) iff $\mathrm{image}(T) = W$ (iff rank $= n$)

proofs

a) $T_2 \circ T_1 (f+g) = T_2(T_1(f+g)) = T_2(T_1(f) + T_1(g)) = T_2(T_1(f)) + T_2(T_1(g))$
$T_2 \circ T_1 (kf) = T_2(T_1(kf)) = T_2(k\, T_1(f)) = k\, T_2(T_1(f))$ ∎

b) $T^{-1}(T(f) + T(g)) = T^{-1}(T(f+g)) = f+g = T^{-1}(T(f)) + T^{-1}(Tg)$
$T^{-1}(k T(f)) = T^{-1}(T(kf)) = kf = k T^{-1}(T(f))$ ∎

c) $V \underset{\text{isomorphic}}{\cong} \mathbb{R}^n$, $\mathbb{R}^n \cong W$, so by (a) $V \cong W$.

In fact, if $\{f_1, f_2, \ldots f_n\}$ is V-basis & $\{g_1, g_2, \ldots g_n\}$ is W-basis then $T(c_1 f_1 + c_2 f_2 + \ldots + c_n f_n) := c_1 g_1 + c_2 g_2 + \ldots + c_n g_n$ is an isomorphism. ∎

d) follows from (a) and $\mathbb{R}^n \ncong \mathbb{R}^m$ for $n \neq m$ ∎

e) use rank + nullity theorem! ∎