Math 2270-3
Monday Oct. 5 §4.1-4.2

- Finish Friday Examples 1 & 2.

Here's another one:

<u>Example 3</u> Let $W \subset M_{2\times 2}$ be give by $W = \{ A_{2\times 2} \text{ s.t. } A\begin{bmatrix}1&2\\0&1\end{bmatrix} = \begin{bmatrix}1&2\\0&1\end{bmatrix}A \}$

- Show W is a subspace of $M_{2\times 2}$
- Find a basis for W.

Notice how in Examples 1, 2, 3, we ultimately ended up solving linear systems of equations to find our solutions... usually these were systems in which we were looking for coordinates with respect to some underlying basis.

Without really thinking about it, we often used

<u>Theorem</u>: Let V be a linear space with basis $\mathcal{B} = \{f_1, f_2, \ldots f_n\}$.
For $f = c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$, consider $[f]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$.
Then

(i) $[f+g]_{\mathcal{B}} = [f]_{\mathcal{B}} + [g]_{\mathcal{B}}$

← because if $f = c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$
& $g = d_1 f_1 + d_2 f_2 + \ldots + d_n f_n$
then $f+g = (c_1+d_1) f_1 + (c_2+d_2) f_2 + \ldots + (c_n+d_n) f_n$

(ii) $[kf]_{\mathcal{B}} = k[f]_{\mathcal{B}}$ for $k \in \mathbb{R}$

... because of our linear space axioms!

and $kf = (kc_1) f_1 + (kc_2) f_2 + \ldots + (kc_n) f_n$.

By using coordinates we prove these generalizations of facts we learned in Chapter 3 for $\mathbb{R}^n$ and its subspaces:

<u>Theorem</u> Let the linear space V have a basis $\mathcal{B} = \{f_1, f_2, \dots, f_n\}$. Then

(a) Fewer than n vectors cannot span V

(b) More than n vectors in V must be linearly dependent.

(c) So every basis of V has exactly n vectors, and $\dim(V) = n$ is well-defined

(d) If $\dim V = n$, then a collection of n independent vectors in V automatically spans, so is a basis

(e) If $\dim V = n$, then a collection of n vectors in V which spans, is automatically independent, so is a basis.

<u>Proof</u>: Let $\mathcal{C} = \{g_1, g_2, \dots g_m\} \subset V$. Each of these facts is obtained by considering the linear combination equation

$$\boxed{(1) \qquad c_1 g_1 + c_2 g_2 + \dots + c_m g_m = f = d_1 f_1 + d_2 f_2 + \dots + d_n f_n}$$

Since $\mathcal{B}$ is a basis, and since vectors are equal iff their coordinates (w.r.t. $\mathcal{B}$) agree,

this equation is equivalent to (by previous thm)

$$c_1 [g_1]_{\mathcal{B}} + c_2 [g_2]_{\mathcal{B}} + \dots + c_m [g_m]_{\mathcal{B}} = [f]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

which is a matrix equation

$$n \left\{ \overbrace{\begin{bmatrix} [g_1]_{\mathcal{B}} & [g_2]_{\mathcal{B}} & \cdots & [g_m]_{\mathcal{B}} \end{bmatrix}}^{m} \right. \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

So (1) is equivalent to

(2) $$\boxed{A_{n \times m} \vec{c} = \vec{d}}$$

a) $m < n \Rightarrow \text{Image}(A) \subset \mathbb{R}^n$ is at most m-dim'l, i.e. not all of $\mathbb{R}^n$; not all $\vec{d}$'s in image, so $\mathcal{C}$ can't span.

b) $m > n$, $\vec{d} = \vec{0} \Rightarrow$ non-trivial sol'ns $\vec{c} \Rightarrow \mathcal{C}$ dependent

c) (a), (b) $\Rightarrow$ c

d) $m = n$. If only sol'n to $A_{n \times n} \vec{c} = \vec{0}$ is $\vec{c} = 0$ then $\text{rref}(A) = I$ and $\text{image}(A) = \mathbb{R}^n$ so $\{g_1, g_2, \dots g_n\}$ span.

e) $m = n$. If $\text{Image}(A) = \mathbb{R}^n$ then $\text{rref}(A) = I$ and only sol'n to $A\vec{c} = \vec{0}$ is $\vec{c} = \vec{0} \Rightarrow \{g_1, g_2, \dots g_n\}$ independent ■

<u>Examples</u>: Is $\{1+x,\ 3x+2-x^2,\ x^2-1,\ 2x+4\}$ a basis for $\mathbb{P}_2$?

Is $\{1+x,\ 2x+3,\ 5x-7\}$ a basis for $\mathbb{P}_1$?

<u>Deeper example</u> (this explains why at least one sort of partial fractions always works!)

Let α, β, γ be three distinct (different) real numbers.
Explain why for given a_0, a_1, a_2 the partial fractions equation

$$\frac{a_0 + a_1 x + a_2 x^2}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}$$

has a unique sol'n $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$.

By writing the RHS over common denominator, this equation is equivalent to

$$a_0 + a_1 x + a_2 x^2 = A \underbrace{(x-\beta)(x-\gamma)}_{f_1(x)} + B \underbrace{(x-\alpha)(x-\gamma)}_{f_2(x)} + C \underbrace{(x-\alpha)(x-\beta)}_{f_3(x)}$$

So result is true iff

$$\mathcal{B} = \{f_1, f_2, f_3\} \text{ is a basis for } \mathbb{P}_2.$$

By previous page, since $\dim(\mathbb{P}_2) = 3$, only need to verify $\{f_1, f_2, f_3\}$ <u>independent</u>

But if

$$c_1 f_1 + c_2 f_2 + c_3 f_3 \equiv 0$$

then

$$\begin{aligned}
@\, x=\alpha: &\quad c_1(\alpha-\beta)(\alpha-\gamma) + 0 + 0 = 0 \Rightarrow c_1 = 0 \\
@\, x=\beta: &\quad 0 + c_2(\beta-\alpha)(\beta-\gamma) + 0 = 0 \Rightarrow c_2 = 0 \\
@\, x=\gamma: &\quad 0 + 0 + c_3(\gamma-\alpha)(\gamma-\beta) = 0 \Rightarrow c_3 = 0
\end{aligned}$$

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You can use these ideas to explain why all the partial fraction decomposition algorithms work. It's linear algebra!
(no wonder they just taught you the algorithm in Calculus!)

§4.2 Linear transformations : Generalizes Chapter 3

<u>Def</u>: Let $T: V \to W$ be a function with domain V and target W, where both V and W are linear spaces.

Then T is called a <u>linear transformation</u> (or "linear") iff

$$\text{(i)}\ T(f+g) = T(f)+T(g) \quad \forall\, f,g \in V$$
$$\text{(ii)}\ T(kf) = k\,T(f) \quad \forall\, f \in V,\ k \in \mathbb{R}.$$

<u>Examples</u>: $T: \mathbb{R}^n \to \mathbb{R}^m$ $T(\vec{x}) = A\vec{x}$ are precisely the linear transformations from $\mathbb{R}^n \to \mathbb{R}^m$.

<u>Coordinates</u>: If $\mathcal{B} = \{f_1, f_2, \ldots f_n\}$ is a basis for the linear space V, then

$T(f) := [f]_{\mathcal{B}}$ is a linear transformation, $T: V \to \mathbb{R}^n$.

Also, $T^{-1}: \mathbb{R}^n \to V$
$$T^{-1}\left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$
is linear.

<u>Derivative operators</u>:

$D: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$

$D(f) := f'$ is linear $\longleftarrow$ these are just two of the easy derivative rules:
$$(f+g)' = f' + g'$$
$$(kf)' = kf'$$

<u>Def</u>: If $T: V \to W$ is linear then

$\ker(T) := \{ f \in V \text{ s.t. } T(f) = 0 \}$

$\text{Image}(T) := \{ g \in W \text{ s.t. } g = T(f) \text{ for some } f \in V \}$

<u>Example</u>: What are the kernel and image of the derivative operator D?

What theorems can you think of, from Chapter 3, that might also apply to general linear transformations?

To be continued!