Math 2270-3
Wednesday Oct 28

Lab on Friday LCB 115 (also is class lecture on §5.5)

§5.4 Least squares approximate sol'ns to $A\vec{x}=b$ & applications

- Finish Tuesday notes, which also include the end of §5.3
  - transpose algebra
  - matrix for projection linear transformation - if you have o.n. basis $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_k\}$ for $W \subset \mathbb{R}^n$
  - Least squares sol'ns, p.3-4 Tuesday.

continue that chain of discussion & examples:

- Suppose $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is a basis for W, not necessarily o.n.
  You can find the matrix for projection without 1st constructing o.n. basis!

from page 4 Tuesday: For $A = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_k\right]$

$$\left.\begin{aligned} & A\vec{x} = \mathrm{proj}_W b \\ \text{iff} \quad & A^T A \vec{x} = A^T b \end{aligned}\right\}$$

notice, from , because col's of A are independent (basis for W)
exactly one $\vec{x}$ solves this system,
therefore the $[A^T A]_{k \times k}$ matrix has an inverse (by our "equivalent to invertible" list)

$$\text{so,} \quad \vec{x} = (A^T A)^{-1} A^T b$$

$$\& \quad \underset{\mathrm{proj}_W \vec{b}}{\overset{\shortparallel}{A\vec{x}}} = \underbrace{\left[A(A^T A)^{-1} A^T\right]}_{\text{so this is the projection matrix!!}} \vec{b}$$

Example 4

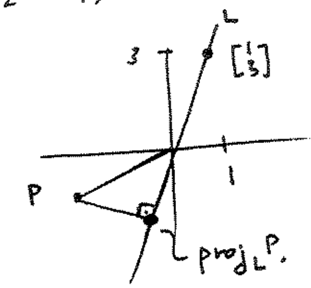
Check our amazing projection matrix formula!

4a) If the columns of A are orthonormal, is this the formula AA^T on p.2 Tuesday? Why?

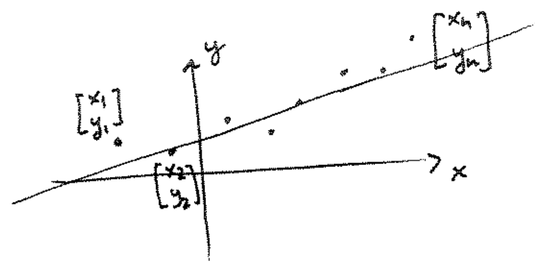
4b) Example 1c): Use $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ to compute the matrix for projecting onto the plane $-x_1 + 2x_2 + x_3 = 0$, yet again. (Use back of this page.)

<u>Example 5</u> : What's the matrix for projecting onto the line $x_2 = 3x_1$, in $\mathbb{R}^2$?

(this seemed so hard in chapter 2 - it should be easy now.)



Applications of least-squares to data fitting.



- find the best line formula
  $y = mx + b$
  to fit n data points $(x_1, y_1) \cdots (x_n, y_n)$

We seek $\begin{bmatrix} m \\ b \end{bmatrix}$ s.t.

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = m \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

i.e. $$\underset{\text{known}}{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix}} \begin{bmatrix} m \\ b \end{bmatrix} = \underset{\text{known.}}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}$$. Not solvable exactly, unless pts actually are on line!

$$A \begin{bmatrix} m \\ b \end{bmatrix} = \vec{y}$$

Least square's soln: $A^T A \begin{bmatrix} m \\ b \end{bmatrix} = A^T \vec{y}$. Unique soln $\begin{bmatrix} m \\ b \end{bmatrix}$ exists (as long as at least 2 x_i-values are different) makes col's A ind.!

You are solving. $A \begin{bmatrix} m \\ b \end{bmatrix} = \text{proj}_W \vec{y}$, $W = \text{span}\left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$

so $$\left\| m \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right\|^2$$ is as small as possible

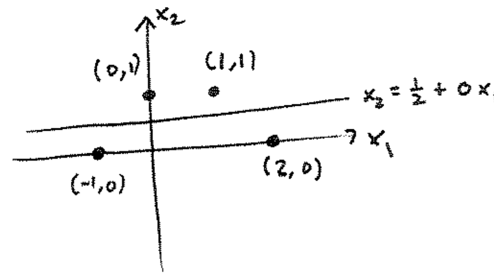
i.e. you've minimized $\sum_{i=1}^{n} (mx_i + b - y_i)^2$,

the sum of the squared vertical deviations between the line $y = mx + b$ & the n data points.

Example 6

6a) Find the least squares line fit for the 4 pts $\{(-1,0),(0,1),(1,1),(2,0)\}$. Sketch

$$m\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}+b\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$



$$\begin{bmatrix}-1&1\\0&1\\1&1\\2&1\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix}=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$ no sol!

$$\begin{bmatrix}-1&0&1&2\\1&1&1&1\end{bmatrix}\begin{bmatrix}-1&1\\0&1\\1&1\\2&1\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix}=\begin{bmatrix}-1&0&1&2\\1&1&1&1\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}6&2\\2&4\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

$m=0$
$b=\frac{1}{2}$!

```
> with(LinearAlgebra):
  with(plots):
```

6b) Find the "best" quadratic fit, same 4 pts! $f(x)=c_0+c_1x+c_2x^2$:

$$c_0\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+c_1\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}+c_2\begin{bmatrix}1\\0\\1\\4\end{bmatrix}=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

1's $\quad x_i$'s $\quad x_i^2 \quad y_i$

```
> CT := Matrix(3, 4, [1, 1, 1, 1, -1, 0, 1, 2, 1, 0, 1, 4]):
  C := Transpose(CT);
```

$$C := \begin{bmatrix}1&-1&1\\1&0&0\\1&1&1\\1&2&4\end{bmatrix}$$

```
> ys := Vector([0, 1., 1, 0]): #made one a decimal, to get dec. rather than frac.
  (CT.C)^(-1).CT.ys; # least squares solution!
```

$$\begin{bmatrix}1.\\0.500000000000000000\\-0.500000000000000000\end{bmatrix}$$

```
> q := t -> 1 + .5·t - .5·t^2:
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> plot1 := plot(q(t), t=-2..3, y=-1..2, color=black):
  plot2 := pointplot({[-1, 0], [0, 1], [1, 1], [2, 0]}):
  display({plot1, plot2}, title='best quadratic fit');
```
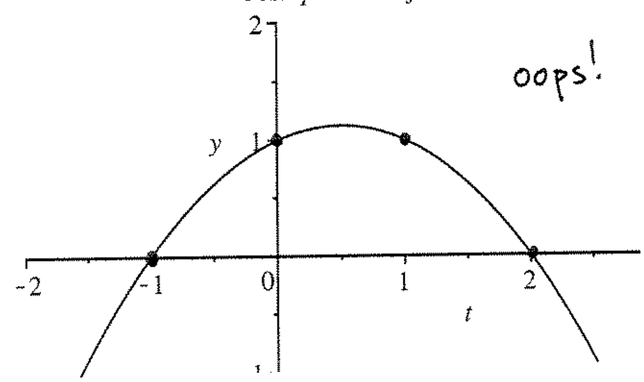

best quadratic fit



oops!