Math 2270-3
Tuesday Oct 27

§5.3 Orthogonal transformations
Matrix transposes
projection formulas and matrices

- Finish Monday notes : pages 2 – 4

Surprising applications for Chapter 5!
Wednesday – §5.4 – in class
Friday – §5.5. – in LCB 115 computer lab.
(project 2)

Then play with transpose algebra :

$(A^T)^T = A$

$(AB)^T = B^T A^T$ (notice order, like for inverse)

check this:

$(A^{-1})^T = (A^T)^{-1}$

check this:

<u>Def</u> : $A = [a_{ij}]$ is <u>symmetric</u> iff $A^T = A$ $(a_{ij} = a_{ji} \ \forall i,j)$

$A = [a_{ij}]$ is <u>antisymmetric</u> iff $A^T = -A$ $(a_{ij} = -a_{ji} \ \forall i,j)$

$$\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

If $A_{m\times n}$ the $A^TA = A^T_{n\times m} A_{m\times n}$ is $n\times n$ symmetric

$AA^T = A_{m\times n} A^T_{n\times m}$ is $m\times m$ symmetric

check this:

<u>example</u> $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute A^TA

AA^T

Matrix for projection onto a subspace $V \subset \mathbb{R}^n$ (uses transpose)

Let $\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\}$ be an orthonormal basis for V

Find the matrix for $T(\vec{x}) = \operatorname{proj}_V \vec{x}$

Not so hard, actually:

$$\operatorname{proj}_V \vec{x} = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2 + \ldots (\vec{x}\cdot\vec{u}_k)\vec{u}_k$$

$$= \left[\vec{u}_1 \middle| \vec{u}_2 \middle| \cdots \vec{u}_k\right] \begin{bmatrix} \vec{x}\cdot\vec{u}_1 \\ \vec{x}\cdot\vec{u}_2 \\ \vdots \\ \vec{x}\cdot\vec{u}_k \end{bmatrix}$$

$$= \left[\vec{u}_1 \middle| \vec{u}_2 \middle| \cdots \vec{u}_k\right] \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vdots \\ \vec{u}_k^T \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} = (AA^T)\vec{x} \qquad \text{for } A = \left[\vec{u}_1 \middle| \vec{u}_2 \middle| \cdots \vec{u}_k\right]$$

Example 1: Let $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ s.t. } -x_1 + 2x_2 + x_3 = 0 \right\}$. Find the matrix for $T(\vec{x}) = \operatorname{proj}_V \vec{x}$

a) Start with $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

G.S. to get $\mathcal{U} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

check:

then find AA^T, see if answer agrees with similar matrix way, on the right.

Check:

b) Chapter 3-4 way:

The matrix B for T with respect to the basis $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{why?}$$

$$S_{E \leftarrow \mathcal{C}} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{why?}$$

$$S_{E\leftarrow\mathcal{C}}\, B\, S_{\mathcal{C}\leftarrow E}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & 5/6 \\ \frac{1}{3} & \frac{1}{3} & -1/3 \\ -\frac{1}{6} & \frac{1}{3} & 1/6 \end{bmatrix}$$

$$= \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix}$$

honest!

§5.4 Least squares approximate solutions to $A\vec{x} = \vec{b}$

In experiments you must often find a solution to

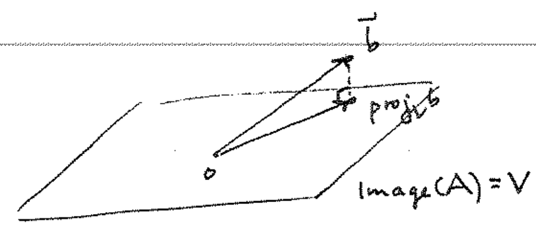
$$A\vec{x} = \vec{b}$$

where no solution exactly exists → because e.g. $\vec{x}$ is a collection of data points.
(we'll see lots of real examples tomorrow.)

the problem for an inconsistent system is that $\vec{b}$ is <u>not</u> in the image of A.

The solution(s) to

$$* \quad A\vec{x} = \mathrm{proj}_{\mathrm{Im}(A)} \vec{b}$$



get $A\vec{x}$ as close to $\vec{b}$ as possible.
they are called "least squares" sol'ns.

<u>Example 2</u>: Find the least-squares sol'n to $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \vec{b}$

↑ "C"

Image(C) = the plane $-x_1 + 2x_2 + x_3 = 0$,
see <u>example 1</u>. $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ is not on this plane!

Using $\mathcal{U} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ basis for Image(C),

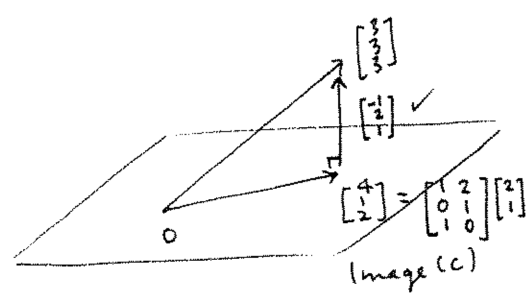
$$\mathrm{proj}_V \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = (\quad) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (\quad) \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$ closest pt. to $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ on image(C)

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{array}\right]$$

$x_1 = 2$
$x_2 = 1$ consistent!

$$\boxed{\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$



There's actually a smart way to find least squares solutions that doesn't require an orthonormal basis:

$$A\vec{x} = \operatorname{proj}_{\operatorname{Im}(A)} \vec{b}$$

$$\Longleftrightarrow \quad \vec{b} - A\vec{x} \in (\operatorname{Im}(A))^{\perp}$$

$\Longleftrightarrow \quad \vec{b} - A\vec{x}$ is $\perp$ to a collection of vectors spanning $\operatorname{Im}(A)$

e.g. the columns of A!

$$\Longleftrightarrow \quad A^T(\vec{b} - A\vec{x}) = \vec{0}$$

the rows of A^T are the columns of A!

$$\Longleftrightarrow \quad A^T\vec{b} - A^TA\vec{x} = \vec{0}$$

$$\Longleftrightarrow \quad A^TA\vec{x} = A^T\vec{b}$$

this system will always be consistent, since least square's sol'ns always exist. If columns of A are independent they are a basis for $\operatorname{Im}(A)$, so sol'n is unique.

Example 3 : redo example 2 w/o orthonormal basis

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

no sol'n.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

least square's sol'n.

multiply out & solve this 2×2 system. answer should agree!

projection matrices without o.n. basis & lots of least squares examples, applications, tomorrow. ~ have a look at §5.4.