Math 2270-3
Monday Oct 26

- Discuss the $A = QR$ decomposition
  that keeps track of the Gram-Schmidt orthogonalization
  (or more precisely, relates the original basis ~ cols of A
  to the G.S. orthonormal one ~ cols of B

this is page 3-4 Friday notes.

- Then begin §5.3 "Orthogonal transformations", which relates to the Q matrices above, in the $Q_{n\times n}$ case.

Def: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is called orthogonal
(we will see that ortho-normal would be a better name, but oh well...)
iff it preserves lengths of line segments, i.e. iff

$$\| T(\vec{x}) \| = \| \vec{x} \| \quad \forall \vec{x} \in \mathbb{R}^n$$

sometimes these transformations are called linear isometries, a better name.

Can we characterize orthogonal transformations $T(\vec{x}) = A\vec{x}$ in terms of the matrix A?

Lemma If $T(\vec{x}) = A\vec{x}$ is orthogonal, then T also preserves the dot product, i.e. $T\vec{x} \cdot T\vec{y} = \vec{x}\cdot\vec{y} \;\; \forall \vec{x},\vec{y}$, and hence also the angle, i.e. $\measuredangle \vec{x},\vec{y} = \measuredangle T\vec{x}, T\vec{y} \;\; \forall \vec{x},\vec{y}$.

[you actually expect this since if triangles get mapped to congruent triangles ("SSS"), then also angles between sides will be preserved]

proof if $\| T\vec{x} \|^2 = \| \vec{x} \|^2 \quad \forall \vec{x} \in \mathbb{R}^n$

then $\| T(\vec{u}+\vec{v}) \|^2 = \| \vec{u}+\vec{v} \|^2 \quad \forall \vec{u},\vec{v} \in \mathbb{R}^n$

$$(T\vec{u}+T\vec{v})\cdot(T\vec{u}+T\vec{v}) = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$

$$\cancel{\|T\vec{u}\|^2} + \cancel{\|T\vec{v}\|^2} + 2\, T\vec{u}\cdot T\vec{v} = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} + 2\,\vec{u}\cdot\vec{v}$$

$$T\vec{u}\cdot T\vec{v} = \vec{u}\cdot\vec{v} \quad \forall \vec{u},\vec{v}.$$

$$\& \quad \frac{T\vec{u}\cdot T\vec{v}}{\|T\vec{u}\|\,\|T\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \quad ■$$

So what orthogonal transformations are there?

Let $T(\vec{x}) = A\vec{x}$ be orthogonal

$$A = \left[T(\vec{e}_1) \,\middle|\, T(\vec{e}_2) \,\middle|\, \cdots \,\middle|\, T(\vec{e}_n) \right]$$

so the columns of A must be unit length (because the $\vec{e}_i$ are & length is preserved) & mutually orthogonal (because $\vec{e}_i \cdot \vec{e}_j = 0$ $i \neq j$)

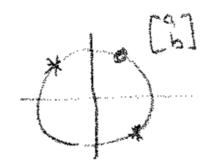
$\rightarrow$ the columns of A must be an orthonormal basis of $\mathbb{R}^n$!

e.g. $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ rotation

$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$ reflection (thru line with $\vec{u} = \begin{bmatrix} \cos\alpha/2 \\ \sin\alpha/2 \end{bmatrix}$)

only ones from $\mathbb{R}^2 \to \mathbb{R}^2$:

$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\quad a^2+b^2=1$, $\quad c^2+d^2=1$, $\quad ac+bd=0$



We just showed $T(\vec{x}) = A\vec{x}$ orthog trans $\Longrightarrow$ cols of A are orthonormal (basis of $\mathbb{R}^n$)

Converse is true too:

Let $A = \left[\vec{w}_1 \,\middle|\, \vec{w}_2 \,\middle|\, \cdots \vec{w}_n \right]$ $\quad \{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_n\}$ orthonormal.

$$\begin{aligned}
A\vec{x} \cdot A\vec{y} &= (x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n) \cdot (y_1\vec{w}_1 + y_2\vec{w}_2 + \cdots + y_n\vec{w}_n) \\
&= \left(\sum_{i=1}^n x_i \vec{w}_i\right) \cdot \left(\sum_{j=1}^n y_j \vec{w}_j\right) \\
&\overset{!}{=} \sum_{i=1}^n \sum_{j=1}^n x_i y_j \underbrace{(\vec{w}_i \cdot \vec{w}_j)}_{\substack{=0 \; j\neq i \\ =1 \; j=i}} \\
&= \sum_{i=1}^n x_i y_i \\
&= \vec{x} \cdot \vec{y}
\end{aligned}$$

so $T(\vec{x}) = A\vec{x}$ preserves dot prod, hence also lengths, so T is orthogonal!

Theorem $T(\vec{x}) = A\vec{x}$ is an <u>orthogonal transformation</u> if and only if the columns of A are orthonormal. We call such an A an orthogonal matrix

<u>Example</u> $A = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ is an orthogonal matrix.

Now I will make a matrix B in which I transpose the columns of A into the rows of B:

$$B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

Compute

$$BA = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

What do you get?

Why???

Def Let $A_{m\times n}$ be a matrix. The transpose matrix A^T is the $n\times m$ matrix defined by

$$\text{entry}_{ij}(A^T) := \text{entry}_{ji}(A)$$

You may obtain the transpose by
(a) transposing the columns of A into the rows of A^T
(b) " " rows of A " " columns of A^T
(c) ~~"reflecting across the diagonal", to get A^T from A~~

e.g. $\text{col}_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ $\longleftrightarrow$ $\text{row}_j(A^T) = [a^T_{j1}, a^T_{j2}, \ldots a^T_{jm}]$
$= [a_{1j}, a_{2j}, \ldots a_{mj}]$

example $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ $\qquad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Theorem A is orthogonal iff $A^T = A^{-1}$

proof:

Corollary: The rows of an orthogonal matrix are also orthonormal!

proof: $AA^T = I$!

e.g. $\begin{bmatrix} 1/\sqrt{14} & 0 & 13/\sqrt{182} \\ 2/\sqrt{14} & -3/\sqrt{13} & -2/\sqrt{182} \\ 3/\sqrt{14} & 2/\sqrt{13} & -3/\sqrt{182} \end{bmatrix}$!