Math 2270-3
Friday Oct 23

no computer lab Monday - we're not ready yet!

HW for Friday Oct 30 ①

5.2: 3, 4, 13, 17, 18, 27, 29, 32, 33, 35, 36, 42

5.3: 1, 2, 7, 11, 15, 19, 21, 27, 31, 35, 40

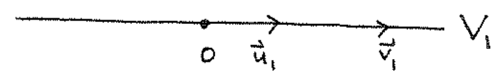
5.4: 1, 2, 35, 21, 22, 23, 25, 31, 32, 39

- finish Wed notes on §5.1
- begin §5.2: How to find orthonormal bases

Here's how! It's an inductive process, called Gram-Schmidt orthogonalization:
Start with a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ for subspace W.

let $V_1 := \text{span}\{\vec{v}_1\}$.

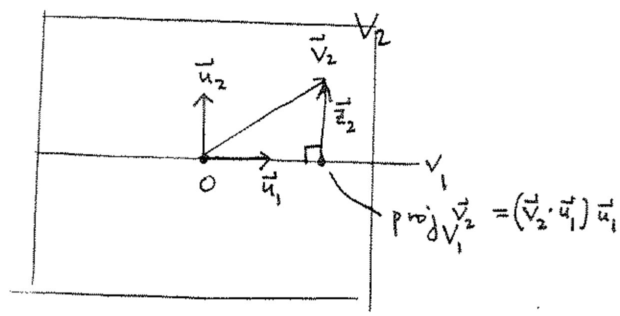
$$\vec{u}_1 := \frac{\vec{v}_1}{\|\vec{v}_1\|}, \quad \text{so} \quad V_1 = \text{span}\{\vec{u}_1\}$$



let $V_2 := \text{span}\{\vec{v}_1, \vec{v}_2\}$

let $\vec{z}_2 = \vec{v}_2 - \text{proj}_{V_1}\vec{v}_2$, so $\vec{z}_2 \perp \vec{u}_1$

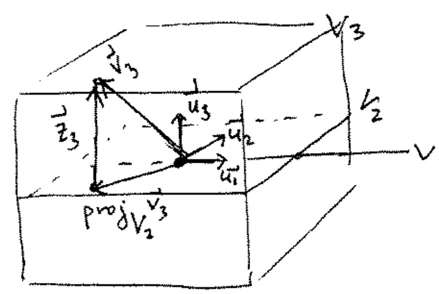
let $u_2 = \frac{z_2}{\|z_2\|}$, so $V_2 = \text{span}\{\vec{u}_1, \vec{u}_2\}$



let $V_3 = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{v}_3\}$

$\vec{z}_3 := \vec{v}_3 - \text{proj}_{V_2}\vec{v}_3$; $z_3 \perp V_2$

$\vec{u}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|}$ so $V_3 = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$



Inductively,

let $V_j = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{j-1}, \vec{v}_j\} = \text{span}\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_{j-1}, \vec{v}_j\}$

$\vec{z}_j = \vec{v}_j - \text{proj}_{V_{j-1}}\vec{v}_j \perp V_{j-1}$. Note, $\vec{z}_j = \vec{v}_j - (\vec{v}_j \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_j \cdot \vec{u}_2)\vec{u}_2 - \ldots - (\vec{v}_j \cdot \vec{u}_{j-1})\vec{u}_{j-1}$

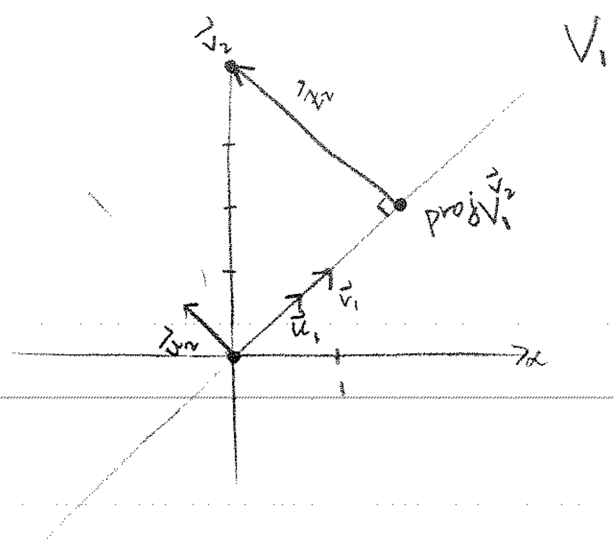
$\vec{u}_j = \frac{\vec{z}_j}{\|\vec{z}_j\|}$

so $V_j = \text{span}\{\underbrace{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_j}_{\text{orthonormal.}}\}$

continue up to $j = k$ ∎

Examples

$$\mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} \right\}$$



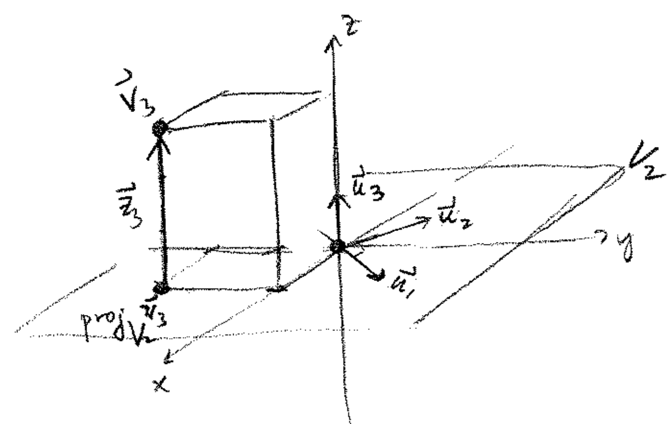
$$\vec{u}_1 := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \vec{z}_2 &= \vec{v}_2 - \text{proj}_{V_1}\vec{v}_2 \\ &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 \\ &= \begin{bmatrix} 0 \\ 4 \end{bmatrix} - \frac{4}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 2 \end{bmatrix} \end{aligned}$$

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

orthonormal basis $\left\{ \underset{\vec{u}_1}{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{\vec{u}_2}{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \right\} = \mathcal{O}$

$$\mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}} \right\}$$



proceeds as first example until step 3

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \vec{z}_3 &= \vec{v}_3 - \text{proj}_{V_2}\vec{v}_3 \\ &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 \\ &= \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \frac{(-1)(1)}{\sqrt{2}\,\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{-3}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \end{aligned}$$

$$\vec{u}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad !$$

orthonormal basis $\left\{ \underset{\vec{u}_1}{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{u}_2}{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{u}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\} = \mathcal{O}$

Gram-Schmidt constructs $\mathcal{O}$ from $\mathcal{B}$

Because $\mathcal{O}$ is orthonormal it is easy to express these two Bases in terms of eachother

Notice $V_j = \text{span}\{\vec v_1, \vec v_2, \ldots \vec v_j\} = \text{span}\{\vec u_1, \vec u_2, \ldots \vec u_j\}$ for each $1 \le j \le k$

So

$$\vec v_1 = (\vec v_1 \cdot \vec u_1)\,\vec u_1$$
$$\vec v_2 = (\vec v_2 \cdot \vec u_1)\,\vec u_1 + (\vec v_2 \cdot \vec u_2)\,\vec u_2$$
$$\vdots$$
$$\vec v_j = (\vec v_j \cdot \vec u_1)\,\vec u_1 + (\vec v_j \cdot \vec u_2)\,\vec u_2 + \ldots + (\vec v_j \cdot \vec u_j)\,\vec u_j$$
$$\vdots$$
$$\vec v_k = \sum_{\ell=1}^{k} (\vec v_k \cdot \vec u_\ell)\,\vec u_\ell$$

notice
$$\vec v_j \cdot \vec u_j = \vec z_j \cdot \vec u_j = \vec z_j \cdot \left(\frac{\vec z_j}{\|\vec z_j\|}\right) = \|\vec z_j\|$$

In matrix form: (column by column)

$\vec u_1 \cdot \vec v_2$

$$* \quad \begin{bmatrix} \vec v_1 & \vec v_2 & \cdots & \vec v_k \end{bmatrix} = \underbrace{\begin{bmatrix} \vec u_1 & \vec u_2 & \cdots & \vec u_k \end{bmatrix}}_{\text{columns are orthonormal}} \underbrace{\begin{bmatrix} \vec v_1\cdot\vec u_1 & \vec v_2\cdot\vec u_1 & \vec u_1\cdot\vec v_3 & & \vec u_1\cdot\vec v_k \\ 0 & \vec v_2\cdot\vec u_2 & \vec u_2\cdot\vec v_3 & & \vec u_2\cdot\vec v_k \\ 0 & 0 & \vec u_3\cdot\vec v_3 & & \vdots \\ \vdots & \vdots & 0 & & \\ & & \vdots & & \vec u_k\cdot\vec v_k \end{bmatrix}}$$

$k \times k$ upper Δular matrix, with positive diagonal entries; it's actually the change of coordinates matrix $S_{\mathcal{O}\leftarrow\mathcal{B}}$ although in this chapter we call it "R"

Thus, and matrix A with k linearly independent col's may be written as

$$A_{m\times k} = \underset{\substack{\uparrow \\ \text{orthonormal columns}}}{Q_{m\times k}}\, R_{k\times k} \quad \text{as above.}$$

<u>Shortcut</u>: If you've found Q you can recover R by multiplying both sides of $*$ on the left by Q^T (Q^T = transpose of Q, turn $\text{col}_j(Q)$ into $\text{row}_j(Q^T)$)

$$\begin{bmatrix} \vec u_1^{\,T} \\ \vec u_2^{\,T} \\ \vdots \\ \vec u_k^{\,T} \end{bmatrix} \begin{bmatrix} \vec v_1 & \vec v_2 & \cdots & \vec v_k \end{bmatrix} = \begin{bmatrix} \vec u_1^{\,T} \\ \vec u_2^{\,T} \\ \vec u_k^{\,T} \end{bmatrix} \begin{bmatrix} \vec u_1 & \vec u_2 & \vec u_k \end{bmatrix} R = IR = R \;!!$$

Examples

$$\mathcal{B} = \left\{ \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} \right\} \qquad \Theta = \left\{ \overset{\vec{u}_1}{\tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \overset{\vec{u}_2}{\tfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \right\}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \vec{v}_1\cdot\vec{u}_1 & \vec{v}_2\cdot\vec{u}_1 \\ 0 & \vec{v}_2\cdot\vec{u}_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{\substack{Q \\ = Rot_{\pi/4}}} \underbrace{\begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}}_{R}$$

check $R = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix}$ ✓

also notice

$$R = \underset{\substack{\uparrow \\ \text{scale}}}{\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}} \underset{\substack{\uparrow \\ \text{shear}}}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\} \qquad \Theta = \left\{ \tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \tfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 4 & -2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} & -1/\sqrt{2} \\ 0 & 2\sqrt{2} & -3/\sqrt{2} \\ 0 & 0 & 3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 4 & -2 \\ 0 & 0 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{2} & 2\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & 2\sqrt{2} & -3/\sqrt{2} \\ 0 & 0 & 3 \end{bmatrix}}_{R}$$

in original notes, 3rd col. was wrong

geometric interpretation:

$T(\vec{x}) = A\vec{x}$ is a composition $T(\vec{x}) = QR\vec{x}$

if $A_{2\times 2}$ then (for non-singular A), it's the composition of

first) ~~scale~~ shear & ~~shear~~ scale (roughly speaking)

second) rotate or reflect.

for $A_{n\times n}$ there is an analogous geometric interpretation.

what happens if the columns of A are not linearly ind?