Math 2270-3
Friday October 2

§4.1 Introduction to linear spaces

HW for Oct 9
4.1 1, 2, 5, 6, 10, 13, 14, 20, 25, 30, 35, 48
4.2 1, 2, 5, 6, 10, 19, 26, 27
4.3 1 2, 3 5 7, 8 13 15 26 29 35 42 49

linear spaces aka linear combination spaces aka vector spaces

Definition A linear space V is a set of objects together with operations "+" and "scalar multiplication ·" so that

$$\alpha)\ f, g \in V \Rightarrow f+g \in V$$
$$\beta)\ f \in V, k \in \mathbb{R} \Rightarrow k \cdot f \in V$$

Furthermore, the following axioms must hold for V to be a linear space: $\forall f, g, h \in V$; $c, k \in \mathbb{R}$

1) $(f+g)+h = f+(g+h)$ — + is Associative

2) $f+g = g+f$ — + is commutative

3) $\exists$ neutral element $n \in V$ s.t. $f+n = f \quad \forall f \in V$ — additive identity
we denote this element by 0 (or $\vec{0}$)

note: neutral element is unique. because if n_1 & n_2 are both neutral then $n_1 = n_1 + n_2 = n_2$ (n_2 neutral, n_1 neutral)

4) $\forall f \in V\ \exists g \in V$ s.t. $f+g = 0$ — additive inverse.
we denote this element g by $-f$

note additive inverse is unique. Since if $f+g_1 = 0$ and $f+g_2 = 0$ then
$$(f+g_1)+g_2 = (f+g_2)+g_1$$
$$0+g_2 = 0+g_1$$
$$g_2 = g_1$$

5) $k(f+g) = kf + kg$

6) $(c+k)f = cf + kf$

7) $c(kf) = (ck)f$

8) $1 \cdot f = f$

consequence of (6):
note $0f = 0$ (0 on the left: scalar; 0 on the right: neutral elt)
because
$$0f = (0+0)f$$
$$0f = 0f + 0f$$
so
$$0f + (-0f) = (0f+0f)+(-0f)$$
$$0 = 0f + (0f + (-0f))$$
$$0 = 0f + 0$$
$$0 = 0f \quad ■$$

note $-f = (-1)\cdot f$
since
$$(1+(-1))f = 0f = 0$$
$$1\cdot f + (-1)\cdot f = 0$$
$$f + (-1)\cdot f = 0.$$
So $(-1)\cdot f$ is additive inverse. ■

Examples!

(A) (i) $\boxed{V = \mathbb{R}^n} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ s.t. } x_i \in \mathbb{R} \quad i = 1, 2, \ldots n \right\}$

with the vector addition & scalar multiplication we are now completely comfortable with.

(ii) and $\boxed{\text{subspace } W \subset \mathbb{R}^n}$ is itself a vector space!
the subspace conditions $(\alpha), (\beta)$ on page 1
(closure under addition & scalar multiplication)
imply that $\vec{0} = 0\vec{w} \in W$
and that for $\vec{w} \in W$, $(-1)\vec{w} = -\vec{w} \in W$.
all other arithmetic properties hold in $\mathbb{R}^n$, so they hold on the subset W as well.

(iii) $\boxed{\text{Spaces of matrices}}$, e.g. $M_{2\times 3} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ s.t. } a_{ij} \in \mathbb{R} \quad \begin{matrix} i = 1..3 \\ j = 1..2 \end{matrix} \right\}$

with the usual matrix addition & scalar multiplication.
(note, this is "the same as" $\mathbb{R}^6$; $M_{m\times n}$ "same as" $\mathbb{R}^{mn}$)
$\downarrow$
$M_{2\times 3}$ in terms of addition & scalar multiplication

symbol for natural #'s i.e. counting #'s.

(iv) $\boxed{\mathbb{R}^{\mathbb{N}}} = \{ \{x_1, x_2, x_3, \ldots\} \text{ s.t. each } x_i \in \mathbb{R} \}$
i.e. the space of sequences. (this is like an infinite version of $\mathbb{R}^n$)

$\{x_1, x_2, x_3, \ldots\} + \{y_1, y_2, y_3, \ldots\} := \{x_1+y_1, x_2+y_2, x_3+y_3, \ldots\}$

What is $\vec{0}$? $\qquad k\{x_1, x_2, \ldots\} := \{kx_1, kx_2, \ldots\}$.

What is $-\vec{x}$?

notice, another way to think of $\mathbb{R}^{\mathbb{N}}$ is as $\{ \underbrace{f : \mathbb{N} \to \mathbb{R}}_{\text{a fcn.}} \}$, where we indentify $f(i)$ with x_i.

i.e. for each natural number i, we associate a real number x_i, in the sequence definition for $\mathbb{R}^{\mathbb{N}}$
This transformation $i \longmapsto x_i$ is just some function f.

this leads to...

B (i) $\mathbb{F} := \{ f: \underset{\text{domain}}{\underset{\uparrow}{\mathbb{R}}} \to \underset{\text{target}}{\underset{\uparrow}{\mathbb{R}}} \text{ s.t. } f \text{ is a function} \}$

e.g. $f(x) := x^2$ is an element of $\mathbb{F}$

$g(x) := \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ is an element of $\mathbb{F}$

We define $f+g$ and kf $(k \in \mathbb{R})$ just as in Calculus:

$$(f+g)(x) := f(x) + g(x)$$
$$(k \cdot f)(x) := k \cdot f(x)$$

it's as if for each $x \in \mathbb{R}$ the function f has "component" $f(x)$.

two functions are equal means $f(x) = g(x)$ $\forall x$.
thus the neutral element 0 is the zero function

$$\underset{\text{function}}{\underset{\uparrow}{0}}(x) = \underset{\text{real\#}}{\underset{\uparrow}{0}} \quad \forall x.$$

Observe: All the required axioms hold so that $\mathbb{F}$ is a (very big) linear space

Definition

Let V be a linear space.
Let $W \subset V$ be a subset closed under addition & scalar multiplication,
(i.e. α), β) on page 1 hold)

Then W is called a subspace

Remark Subspaces of linear spaces are themselves linear spaces.
(Since $f \in W \Rightarrow 0 = 0f \in W$ and $-f = (-1)f \in W$, and all other properties hold already in the larger space!).

$\mathbb{F}$ has lots of interesting subspaces: See next page!

$\mathbb{F}$ has lots of interesting subspaces, including these:

$\mathbb{F}$

$\cup$

$C(\mathbb{R},\mathbb{R}) := \{ f:\mathbb{R}\to\mathbb{R} \text{ s.t. } f \text{ is continuous} \}$ ⟵ You proved in Calculus that if f and g are cont then so is $f+g$ and if $k \in \mathbb{R}$, then so is kf (followed from limit thms)

$\cup$

$C^1(\mathbb{R},\mathbb{R}) := \{ f:\mathbb{R}\to\mathbb{R} \text{ s.t. } f'(x) \text{ exists } \forall x \text{ and the function } f' \text{ is continuous} \}$ ⟵ follows from $(f+g)' = f'+g'$, $(kf)' = kf'$

$\cup$

$C^2(\mathbb{R},\mathbb{R}) := \{ f:\mathbb{R}\to\mathbb{R} \text{ s.t. } f', f'' \in C(\mathbb{R},\mathbb{R}) \}$

$\cup$

$C^3(\mathbb{R},\mathbb{R})$

$\cup$

$\vdots$

$\cup$

$C^\infty(\mathbb{R},\mathbb{R}) := \{ f:\mathbb{R}\to\mathbb{R} \text{ s.t. } f^{(k)}(x) \in C(\mathbb{R},\mathbb{R}) \ \forall k \in \mathbb{N} \}$ ∞'ly differentiable fcns

$\cup$

$\mathbb{P} := \{ \text{polynomials } p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \text{ for some } n \in \mathbb{N} \text{ and } a_n, a_{n-1}, \dots a_0 \in \mathbb{R} \}$ (of arbitrary degree n)

$\cup$

$\vdots$

$\cup$

$\mathbb{P}_n := \{ \text{polynomials } p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 , \ a_0, a_1, \dots a_n \in \mathbb{R} \}$

i.e. polynomials of degree $\le n$.

$\cup$

$\vdots$

Let $n \in \mathbb{N}$.

Question: Why aren't the polynomials of degree <u>exactly</u> n a subspace?

Hint: check condition (α).

$\cup$

$\mathbb{P}_3$

$\cup$

$\mathbb{P}_2 := \{ p_2(x) = a_2 x^2 + a_1 x + a_0 \text{ s.t. } a_0, a_1, a_2 \in \mathbb{R} \}$

$\cup$

$\mathbb{P}_1$

$\cup$

$\mathbb{P}_0$ "=" $\mathbb{R}$.

Fill in the following definitions!

Let V be a linear space, with $f_1, f_2, .. f_n \in V$

- a linear combination of $f_1, f_2, .. f_n$ is

- the span of $\{f_1, f_2, .. f_n\}$ is

- $\{f_1, f_2 .. f_n\}$ is linearly dependent iff
  linearly independent iff

- $\{f_1, f_2, .. f_n\}$ is a basis for V iff

- If $\{f_1, f_2, .. f_n\}$ is a basis $\mathcal{B}$ for V, and if $f \in \mathcal{B}$, with $f = c_1 f_1 + c_2 f_2 + .. + c_n f_n$ c_j const
  then $[f]_{\mathcal{B}} =$

- $\dim V =$

i.e. $f_1(x) \equiv 1$
$f_2(x) = x$
$f_3(x) = x^2$

Example 1: Show the functions $\{1, x, x^2\}$ are a basis for $\mathbb{P}_2$

- span Let $p(x) \in \mathbb{P}_2$. Then $p(x) = a_2 x^2 + a_1 x + a_0$ by definition.
  = a linear combo of our three fns!

- independent: let $c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 = 0$ ← the zero fn, i.e. $= 0\ \forall x$.

$D_x: \Rightarrow$ $c_2 + 2c_3 x \equiv 0$ (we write $\equiv 0$ to mean zero for all x)

D_x^2 $2c_3 \equiv 0$

$\Rightarrow c_3 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 0$ ∎

(Corollary: $c_1 + c_2 x + c_3 x^2 = d_1 + d_2 x + d_3 x^2$ iff $c_1 = d_1$, $c_2 = d_2$, $c_3 = d_3$
(as functions)

more generally, two polynomials are equal as functions (i.e. $\forall x$) iff all of their corresponding coefficients agree.)

Example 1 con't

- What is the coordinate vector
$$[2+3x-4x^2]_{\mathcal{B}} \qquad \text{for } \mathcal{B} = \{1, x, x^2\} \ ?$$

- Is $\underset{g_1(x),\ g_2(x),\ g_3(x)}{\{x-1,\ x^2+x,\ x^2+1\}}^{=\mathcal{C}}$ a basis for $\mathbb{P}_2$? Hint: No!

- Show $\underbrace{\{x+1,\ x^2+x,\ x^2+1\}}_{=\mathcal{D}}$ is a basis for $\mathbb{P}_2$

- Find $[x]_{\mathcal{D}}$. this means the function $f(x)=x$

Example 2

2a) Show $W := \{p(x) \in \mathbb{P}_2 \text{ s.t. } p(3)=1\}$ is not a subspace of $\mathbb{P}_2$

2b) Show $W := \{p(x) \in \mathbb{P}_2 \text{ s.t. } p(3)=0\}$ is a subspace of $\mathbb{P}_2$. Find a basis!