Math 2270-3
Additional page for Monday Oct 19

Remark on change of basis matrices:

Let $\mathcal{A} = \{g_1, g_2, \ldots g_n\}$
$\mathcal{B} = \{f_1, f_2, \ldots f_n\}$ be two bases for the linear space V.

Then $\left[c_1 g_1 + c_2 g_2 + \cdots + c_n g_n\right]_{\mathcal{B}} = c_1 [g_1]_{\mathcal{B}} + c_2 [g_2]_{\mathcal{B}} + \cdots + c_n [g_n]_{\mathcal{B}}$

$$= \left[[g_1]_{\mathcal{B}} \vdots [g_2]_{\mathcal{B}} \vdots \cdots \vdots [g_n]_{\mathcal{B}} \right] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

So the change of coordinates matrix

$$S_{\mathcal{B} \leftarrow \mathcal{A}} = \left[[g_1]_{\mathcal{B}} \middle| [g_2]_{\mathcal{B}} \middle| \cdots [g_n]_{\mathcal{B}} \right] \quad ; \qquad col_j(S) = [g_j]_{\mathcal{B}}.$$

<u>Example</u> $\mathcal{B} = \{1, x, x^2\}$, $\mathcal{A} = \{1+x, 1-x, x^2\}$ basis for $\mathbb{P}_2$
$\quad f_1, f_2, f_3 \qquad\qquad g_1, g_2, g_3$

$$S := S_{\mathcal{B} \leftarrow \mathcal{A}} = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ [g_1]_{\mathcal{B}} \ [g_2]_{\mathcal{B}} \ [g_3]_{\mathcal{B}} \end{array} \qquad S^{-1} = S_{\mathcal{A} \leftarrow \mathcal{B}} = \begin{array}{c} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ [f_1]_{\mathcal{A}} \ [f_2]_{\mathcal{A}} \ [f_3]_{\mathcal{A}}. \end{array}$$

<u>Example cont'd</u>

$\mathcal{A} = \{1+x, 1-x, x^2\}$

$\mathcal{B} = \{1, x, x^2\}$

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[rrr, "A"] \arrow[dr] \arrow[dd, "S"'] & & & \mathbb{R}^3 \arrow[dd, "S"] \\
& \mathbb{P}_2 \arrow[r, "D"] & \mathbb{P}_2 \arrow[ur] \arrow[dr, "L_{\mathcal{B}}"] & \\
\mathbb{R}^3 \arrow[ur, "L_{\mathcal{B}}^{-1}"'] \arrow[rrr, "B"'] & & & \mathbb{R}^3
\end{tikzcd}$$

$$A = \begin{array}{c} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ [D(1+x)]_{\mathcal{A}} \ [D(1-x)]_{\mathcal{A}} \ [D(x^2)]_{\mathcal{A}} \end{array}$$

checked
$SA = BS$
i.e.
$A = S^{-1} B S$
(alternate way to get A from B, if direct column by column method is harder)

$$B = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\ [D(1)]_{\mathcal{B}} \ [D(x)]_{\mathcal{B}} \ [D(x^2)]_{\mathcal{B}} \end{array}$$