Math 2270-3
Monday Oct. 19

- Finish § 4.3: quickly review page 2 Friday, then complete pages 3-4, to make sure we understand what the matrix of a linear transformation with respect to a basis is; how to compute it; how to use it.

Additional HW for Friday Oct 24:

Chptr 4 TF Review problems p. 184 $\boxed{\text{all multiples of 4 } (4, 8, \ldots)}$

if true, justify. if false, give counterexample

(you also have § 4.3, § 5.1 problems!)

Change of basis – there's room on page 4 Friday to check that changing bases leads to similar matrices.

Experiment with this for Example 1: $D: \mathbb{P}_2 \to \mathbb{P}_2$

$D(p(x)) := p'(x)$

For $\mathcal{B} = \{1, x, x^2\}$

$$B = [D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

For $\mathcal{A} = \{1+x, 1-x, x^2\}$

- Find $A = [D]_{\mathcal{A}}$

- Verify that A and B are similar, by the appropriate change of basis matrices

Chapter 5: Orthogonality in $\mathbb{R}^n$ and beyond!

Begin §5.1: orthogonal bases & projection

Recall from Appendix A, earlier this semester:

$\vec{v}\cdot\vec{w} = \sum_{i=1}^{n} v_i w_i$

a) for $\vec{v},\vec{w}\in\mathbb{R}^n$, $\vec{v}\perp\vec{w}$ iff $\vec{v}\cdot\vec{w}=0$

$\perp$ "perpendicular", "orthogonal"

b) $\|\vec{v}\| := \left(\sum_{i=1}^{n} v_i^2\right)^{1/2} = \sqrt{\vec{v}\cdot\vec{v}}$ "magnitude"

c) $\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ iff $\vec{v}\perp\vec{w}$ Pythagorean Thm.

d) $\vec{u}\in\mathbb{R}^n$ is a unit vector iff $\|\vec{u}\|=1$

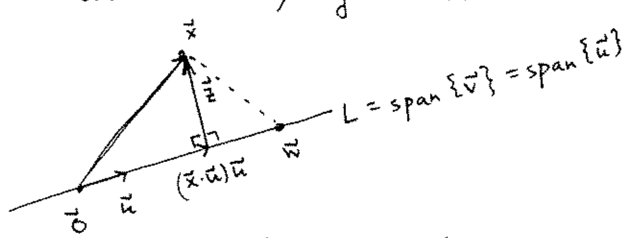
e) If $\vec{v}\in\mathbb{R}^n$, $\vec{v}\neq\vec{0}$, then $\vec{u} := \frac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector in dir. of $\vec{v}$ because $\|t\vec{v}\| = |t|\,\|\vec{v}\|$.

f) If $L=\text{span}\{\vec{v}\}$ is a line thru $\vec{0}$, then for $\vec{x}\in\mathbb{R}^n$,

$\text{proj}_L\vec{x} := (\vec{x}\cdot\vec{u})\vec{u}$ (for $\vec{u} := \frac{1}{\|\vec{v}\|}\vec{v}$)

is the nearest point to $\vec{x}$ on L.

reason is $\mathbb{R}^n$ Pythag. theorem:



define $\vec{z} := \vec{x} - (\vec{x}\cdot\vec{u})\vec{u}$

then $\vec{z}\cdot\vec{u} = (\vec{x} - (\vec{x}\cdot\vec{u})\vec{u})\cdot\vec{u} = \vec{x}\cdot\vec{u} - \vec{x}\cdot\vec{u} = 0$

so $\vec{z}\cdot t\vec{u} = 0 \ \forall t$

so for $\vec{w}\in L$, $\|\vec{w}-\vec{x}\|^2 = \|\vec{w} - (\vec{x}\cdot\vec{u})\vec{u}\|^2 + \|\vec{z}\|^2$

$\geq \|\vec{z}\|^2$.

$=$ iff $\vec{w} = \text{proj}_L\vec{x}$.

Example 1 : Let $L = \text{span}\left\{\begin{bmatrix}2\\1\\1\end{bmatrix}\right\}$

Find $\text{proj}_L \begin{bmatrix}5\\2\\0\end{bmatrix}$

Verify Pythagorean Thm.

These ideas generalize to subspaces of dimension > 1 :

Definition The set $\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\}$ in $\mathbb{R}^n$ is called orthonormal

iff

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

mutually orthogonal

normalized to unit length

Examples we know

1. Standard basis $\{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$ for $\mathbb{R}^n$ (or, e.g. just the first k of them $\{\vec{e}_1, \vec{e}_2, .. \vec{e}_k\}$)
2. rotated bases in $\mathbb{R}^2$, $\left\{ \underbrace{\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}}_{\vec{u}_1}, \underbrace{\begin{bmatrix}-\sin\alpha\\ \cos\alpha\end{bmatrix}}_{\vec{u}_2} \right\}$

Why orthonormal collections are good

Theorem : Let $\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\} \subset \mathbb{R}^n$ be orthonormal. Let $W := \text{span}\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\}$.

Then

a) $\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\}$ are linearly independent, so are a basis ("$\mathcal{B}$") for W

b) for $\vec{v} \in W$, $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}\vec{u}_1 \cdot \vec{v}\\ \vec{u}_2 \cdot \vec{v}\\ \vdots \\ \vec{u}_k \cdot \vec{v}\end{bmatrix}$ (immediately computable!)

c) for $\vec{x} \in \mathbb{R}^n$, there is a unique nearest point to $\vec{x}$ in W, which we call $\text{proj}_W \vec{x}$. The formula for this projection is

$$\text{proj}_W \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + \cdots + (\vec{u}_k \cdot \vec{x})\vec{u}_k$$

Example 2 : Check this theorem for $\{\vec{e}_1, \vec{e}_2, .. \vec{e}_k\} \subset \mathbb{R}^n$

Example 3 : Consider the plane $W \subset \mathbb{R}^3$, $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ s.t. } 2x_1 - 5x_2 + x_3 = 0 \right\}$

a) Verify $\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\} := \mathcal{B}$ is an orthonormal basis for W

Hey, how'd we find this ?! (§5.2)

b) Show $\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} \in W$

c) Find $\left[\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} \right]_{\mathcal{B}}$, using Thm. Check work!

ans $\begin{bmatrix} 12/\sqrt{6} \\ \sqrt{5} \end{bmatrix}$

d) Find $\text{proj}_W \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$

then figure out how I constructed this example to connect with (b)(c)

Example 4 a) Check that $\mathcal{B} := \left\{ \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}$

is an orthonormal basis for $\mathbb{R}^3$

b) For $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $[\vec{x}]_{\mathcal{B}}$ and check answer.

ans $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

Proof of "good" theorem, page 3: $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\} \subset \mathbb{R}^n$ is orthonormal

a) let $\underbrace{c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k}_{} = \vec{0}$

$\Rightarrow (\quad)\cdot\vec{u}_j = \vec{0}\cdot\vec{u}_j = 0$ any $1 \le j \le k$

$\Rightarrow c_j = 0$ each $1 \le j \le k$ ∎ Hence $\mathcal{B}$ is basis for W.

b) Since $\vec{v} \in W$ we may uniquely write

$$\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k \qquad [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}.$$

$$\Rightarrow \vec{v}\cdot\vec{u}_j = (\qquad)\cdot\vec{u}_j$$
$$= c_1(\vec{u}_1\cdot\vec{u}_j) + c_2(\vec{u}_2\cdot\vec{u}_j) + \cdots + c_k(\vec{u}_k\cdot\vec{u}_j)$$
$$= c_j \quad ∎$$

because $\vec{u}_i\cdot\vec{u}_j = \begin{cases} 1 & i=j \\ 0 & i\ne j \end{cases}$!

c) temporarily write

$$\widetilde{\mathrm{proj}}_W\vec{x} := (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2 + \cdots + (\vec{x}\cdot\vec{u}_k)\vec{u}_k$$

we'll show it's the unique nearest point in W to $\vec{x}$, so that we can rewrite it as $\mathrm{proj}_W\vec{x}$.

Define $\vec{z} := \vec{x} - \widetilde{\mathrm{proj}}_W\vec{x}$

Then $\vec{z}\cdot\vec{u}_j = (\vec{x} - \widetilde{\mathrm{proj}}_W\vec{x})\cdot\vec{u}_j \qquad \forall\ 1 \le j \le k$

$$= \vec{x}\cdot\vec{u}_j - (\qquad)\cdot\vec{u}_j$$
$$= \vec{x}\cdot\vec{u}_j - \vec{x}\cdot\vec{u}_j \qquad \text{(as in (b)!)}$$
$$= 0 \qquad \forall\ 1 \le j \le k$$

Since $W = \mathrm{span}\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\}$

deduce $\vec{z} \perp$ every vector in W (any $t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots + t_k\vec{u}_k$)

Let $\vec{w} \in W$.

Then $\|\vec{x} - \vec{w}\|^2 = \| \underbrace{x - \widetilde{\mathrm{proj}}\,\vec{x}}_{\vec{z}} + \underbrace{(\widetilde{\mathrm{proj}}\,\vec{x} - \vec{w})}_{\in W} \|^2$

$$= \|\vec{z}\|^2 + \|\widetilde{\mathrm{proj}}\,\vec{x} - \vec{w}\|^2 \qquad (\text{since } z \perp W)$$

$\ge \|\vec{z}\|^2$, with equality iff $\vec{w} = \widetilde{\mathrm{proj}}\,\vec{x}$!

∎