Math 2270-3
Monday Nov. 9

HW: Maple project is due next Tuesday Nov 17 @ 5 pm.

For this Friday Nov 13
6.1 9, 10 17 32 42 43 48 49
6.2 1, 2, 9 11, 15, 16, 17 30, 31, 32 35 37 38 44

§ 6.1-6.2 determinants algebra.

for $A_{n\times n}$, $\det(A)$ (also written $|A|$) is a scalar which determines whether A^{-1} exists ($|A| \neq 0$ iff A^{-1} exists). the precise definition is based on $\pm$ products of matrix entries, and their sum. Each product contains exactly one entry from each col & row of A, and the $+$ or $-$ is determined by whether there are an even or odd # of "inversions" in the product pattern!

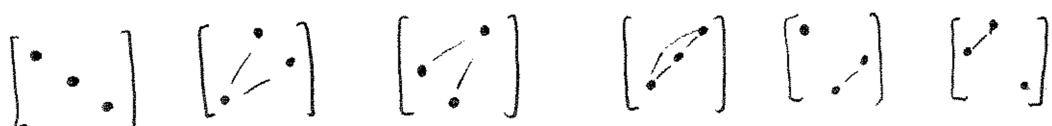
a pair of locations in a pattern for which the ~~entry~~ location in later (lower) row is in earlier column.

(i_1, j_1)
(i_2, j_2) s.t. $i_2 > i_1$, $j_2 < j_1$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

0 inversions 1 inversion

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$



inversions 0 2 2 3 1 1

<u>Definition</u> A <u>pattern</u> P in an $n\times n$ matrix is a collection of n entries chosen so that there is exactly one entry in each row and column.

For a given pattern you can organize the entries by rows: $a_{1j_1}, a_{2j_2}, \cdots a_{nj_n}$ so $(j_1, j_2, \ldots j_n)$ is a permutation of the numbers $(1, 2, \ldots n)$

or by columns: $a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}$ $(i_1, i_2, \ldots i_n)$ permutation of $(1, 2, \ldots n)$

<u>Exercise 1</u>: Explain why $A_{n\times n}$ has $n!$ $(= n(n-1)\cdots 3\cdot 2\cdot 1)$ patterns.

Definition An inversion in a pattern is a pair of distinct entries so that the entry in the later row is in an earlier column (or equivalently, the entry in the later column is in an earlier row)

Definition The signature (or sign) of a pattern P is defined by

$$\operatorname{sgn}(P) = \begin{cases} +1 & \text{if } P \text{ has even \# of inversions} \\ -1 & \text{if } P \text{ has odd \# of inversions.} \end{cases}$$

Def Let $A_{n \times n}$. Then

$$|A| := \sum_{(\text{all})\ P} \underbrace{\operatorname{sgn}(P)}_{\pm,\ \text{as above}}\ \underbrace{\operatorname{prod}(P)}_{\text{the product of all entries in the pattern.}}$$

Exercise 2 Find $|A|$:

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & -1 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 3 Find $|A|$:

$$A = \begin{bmatrix} 6 & 0 & 1 & 0 & 0 \\ 9 & 3 & 2 & 5 & 7 \\ 8 & 0 & 3 & 2 & 9 \\ 0 & 0 & 4 & 0 & 0 \\ 5 & 0 & 5 & 0 & 1 \end{bmatrix}$$

Exercise 4 Find $|A|$:

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 6 \\ 0 & 7 & 6 & 385 & 52 \\ 0 & 0 & -3 & 4 & e \\ 0 & 0 & 0 & 2 & \pi \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 5 Find $|A|$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 & 0 & 0 \\ 3 & 8 & 6 & 0 & 0 & 0 \\ 4 & 9 & 5 & 2 & 1 & 0 \\ 5 & 8 & 4 & 0 & 2 & 5 \\ 6 & 7 & 3 & 0 & 3 & 6 \end{bmatrix}$$

Theorem 1: If A is upper triangular, $A = \begin{bmatrix} a_{11} & & \text{stuff} \\ & a_{22} & \\ 0 & & \ddots \; a_{nn} \end{bmatrix}$, $|A| = a_{11}a_{22} \cdots a_{nn}$

If A is lower triangular, $A = \begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ \text{stuff} & & \ddots \; a_{nn} \end{bmatrix}$, $|A| = a_{11}a_{22} \cdots a_{nn}$

It's also true (as in exercise 5), that if M has a block structure, with one of the off-diagonal ~~sq~~ blocks zero, then:

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \Rightarrow |M| = |A||C|$$

A, C square
(not nec. same size)

$$M = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \Rightarrow |M| = |A||C|$$

$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ not generally true
$|M| = |A||D| - |C||B|$

See p. 258-259, or try to see "why" yourself.

Theorem 2 : $\det(A) = \det(A^T)$

proof when you transpose A into A^T
pattern P is transposed into P^T; $\text{prod}(P)$ in $A = \text{prod}(P^T)$ in A^T;
inversion pairs in P transpose to inversion pairs in P^T, and vise versa,
so $\text{sgn}(P) = \text{sgn}(P^T)$.

example

$$A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ -1 & 2 & -2 & 1 \\ 6 & 1 & -3 & 1 \\ 5 & 0 & -4 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & -1 & 6 & 5 \\ 3 & 2 & 1 & 0 \\ -1 & -2 & -3 & -4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

P $\qquad\qquad$ P^T

$$\text{prod}(P) \text{ in } A = \text{prod}(P^T) \text{ in } A^T$$
$$\text{sgn}(P) \text{ in } A = \text{sgn}(P^T) \text{ in } A^T$$

$$\text{so} \quad \sum_{\substack{P \text{ pattern} \\ \text{in } A}} \text{sgn}(P)\,\text{prod}(P) = \sum_{\substack{P^T \text{ pattern} \\ \text{in } A^T}} \text{sgn}(P^T)\,\text{prod}(P^T)$$

Tomorrow we'll study how the determinant changes if we do elementary row operations (or by Thm 2, elementary column operations).

This will give us an efficient way to compute determinants for general $A_{n\times n}$, as well as letting us figure out why $|A| \neq 0$ iff A^{-1} exist.

& more!

2270-3 2nd midterm
score distribution
n=32