Math 2270-3
Tuesday Nov. 3
§ 5.5 : Inner product spaces ~ review of chapter 5.

Here's the flow chart of how we developed dot product concepts in $\mathbb{R}^n$:

$\mathbb{R}^n$ dot prod. $\vec{x}\cdot\vec{y} := \sum_{i=1}^{n} x_i y_i$

algebra

a) $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$ (symmetry)

b) $\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$
$\vec{x}\cdot(k\vec{y}) = k\ \vec{x}\cdot\vec{y}$ (linear in each factor)

c) $\vec{x}\cdot\vec{x} \geq 0$; $\vec{x}\cdot\vec{x} = 0$ iff $\vec{x} = \vec{0}$ (positive)

from algebra...

magnitude

orthogonal

distance from $\vec{x}$ to $\vec{y}$ $= \|\vec{x}-\vec{y}\|$

Pythagorean Theorem

orthonormal basis's for $V \subset \mathbb{R}^n$, coords for such bases (def.)

Gram-Schmidt construction

projection: $\mathbb{R}^n = V \oplus V^{\perp}$
i.e. $\vec{x} = \underset{\in V}{\mathrm{proj}_V \vec{x}} + \underset{\in V^{\perp}}{\vec{z}}$, uniquely

$\mathrm{proj}_{\vec{x}}\vec{y} \longrightarrow$ Cauchy-Schwarz $\longrightarrow \measuredangle \vec{x},\vec{y}$

Δ inequality

nearest point to $\vec{b}$ in V is $\mathrm{proj}_V \vec{b}$

projection matrices

least squares solns to $A\vec{x} = \vec{b}$

data fitting $\leftrightarrow$ linear regression $\overset{\ln\text{-}\ln}{\longleftrightarrow}$ power laws

polynomial fits

An <u>inner product space</u> is a linear space V together with an inner product $\langle \cdot , \cdot \rangle$ which gives a real number for each pair of vectors, s.t. the following axioms hold:
$(\forall f, g, h \in V, k \in \mathbb{R})$

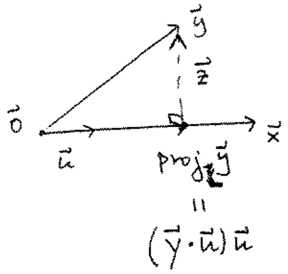
a) $\langle f, g\rangle = \langle g, f\rangle$ (symmetry)

b) $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$
$\langle f, kg\rangle = k\langle f, g\rangle$
(linear in each factor)

c) $\langle f, f\rangle \geq 0$. $\langle f, f\rangle = 0$ iff $f = 0$

From these algebra axioms, the entire concept chart at the left also holds!
(except maybe least squares solns in case $\dim V = \infty$)

Cauchy-Schwarz details:
(we ran out of time in a previous lecture)



$L = \text{span}\{\vec{x}\} = \text{span}\{\vec{u}\}$

$\vec{u} := \dfrac{\vec{x}}{\|\vec{x}\|}$

$\vec{z} := \vec{y} - (\vec{y}\cdot\vec{u})\vec{u}$

$\vec{z}\cdot\vec{u} = 0$

so $\|(\vec{y}\cdot\vec{u})\vec{u}\|^2 + \|\vec{z}\|^2 = \|\vec{y}\|^2$ (Pyth.)

$|\vec{y}\cdot\frac{\vec{x}}{\|\vec{x}\|}|^2 \leq \|y\|^2$

C.S.: $|\vec{y}\cdot\vec{x}| \leq \|\vec{y}\|\|\vec{x}\|$

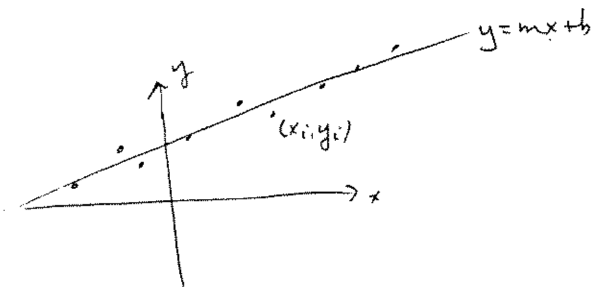
so $\cos\theta := \dfrac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|}$ is consistently defined.
$-1 \leq \cos\theta \leq 1$.

and Δ ineq. holds

$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

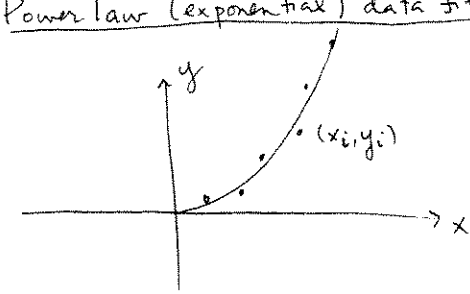
Linear regression for n data points $\{(x_i, y_i)\}_{i=1\ldots n}$ is least squares sol'n $\begin{bmatrix} m \\ b \end{bmatrix}$ to

$$m\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$



Precisely, it's the line $y = mx + b$.

Power law (exponential) data fit seek fcn $y = bx^m$ to fit data

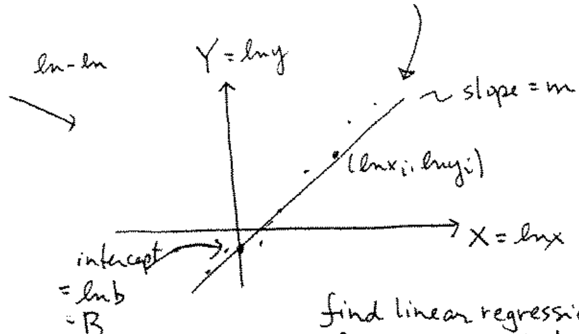


$\boxed{\text{See Friday notes example}}$

and part A of project 2

iff $\ln y = \ln b + m \ln x$
$Y = B + mX$

$Y := \ln y$
$X := \ln x$



find linear regression line for ln-ln data!

examples of function space inner products

$V = \{ f : [a,b] \to \mathbb{R} \text{ s.t. } f \text{ is continuous} \} := C([a,b])$

$\langle f, g \rangle := \int_a^b f(t)g(t)\,dt$ (or some positive multiple of this integral)

Check a) b) c) algebra

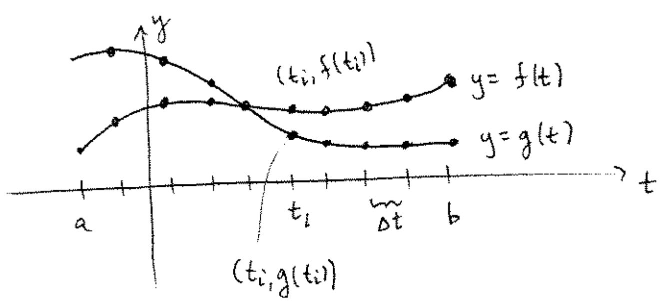
$\langle f,g \rangle$ is not so different from $\mathbb{R}^n$ dot product if you think of Riemann sums:

let $\Delta t = \frac{b-a}{n}$

$t_i = a + i\Delta t \quad i = 1, 2, \ldots n$

$$\underset{\substack{\uparrow \\ \text{function} \\ \text{inner product}}}{\langle f, g \rangle} = \int_a^b f(t)g(t)\,dt = \lim_{n \to \infty} \Big(\underbrace{\sum_{i=1}^n f(t_i)g(t_i)}_{\begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{bmatrix} \cdot \begin{bmatrix} g(t_1) \\ g(t_2) \\ \vdots \\ g(t_n) \end{bmatrix}} \Big) \Delta t$$

$\mathbb{R}^n$ dot product!



this is the set up in Maple project 2B

<u>Example 1</u> $V = C([-1,1])$; $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)\,dt$ $\rightsquigarrow$ you Gram-Schmidt $\{1, t, t^2, t^3, \ldots\}$ and compute projections

<u>Example 2</u> $V = C([-\pi, \pi])$; $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)\,dt$ $\rightsquigarrow \{ \frac{1}{\sqrt{2}}, \cos t, \sin t, \cos 2t, \sin 2t, \ldots \}$ are orthonormal !!

If there's time, we can work out parts of example 1 & example 2
by hand → we did a lot of this in the lab yesterday, with comp. help.

example 1 : G.S. $\{1, t, t^2\} \subset C([-1,1])$ with $\langle f,g\rangle := \int_{-1}^{1} f(t)g(t)\,dt$

in your project you're supposed to G.S. $\{1, t, t^2, t^3\}$ and project "$\sin t$" onto this 4-dim'l space

example 2 : Show $\{\frac{1}{\sqrt{2}}, \cos t, \cos 2t, \ldots \cos Nt, \sin t, \sin 2t, \ldots \sin Nt\}$ is an orthonormal basis for its span, a subspace of $C([-\pi,\pi])$ "V_n"
with $\langle f,g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt$

compute proj_{V_n} "t" (the order n Fourier sum for $f(t)=t$)

(in project, you're supposed to compute $\text{proj}_{V_n} |t|$.)

$$\text{proj}_{V_n} f = \frac{a_0}{2} + \sum_{j=1}^{n} a_j \cos jt + \sum_{j=1}^{n} b_j \sin jt$$

$$\frac{a_0}{2} = \langle f, \tfrac{1}{\sqrt{2}}\rangle \tfrac{1}{\sqrt{2}} = \frac{1}{2}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$j \geq 1$ $\quad a_j = \langle f, \cos jt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jt\,dt$

$$b_j = \langle f, \sin jt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jt\,dt$$

For orthonormal properties, trig is nice:

$$\cos(mt)\cos(nt) = \tfrac{1}{2}\left[\cos(m+n)t + \cos(m-n)t\right] \qquad \left(\text{if } m=n, \text{ get } \tfrac{1}{2}\left[\cos 2nt + 1\right]\right)$$

$$\sin(mt)\sin(nt) = \tfrac{1}{2}\left[-\cos(m+n)t + \cos(m-n)t\right] \qquad \left(\text{if } m=n, \text{ get } \tfrac{1}{2}\left[-\cos 2nt + 1\right]\right)$$

$$\cos(mt)\sin(nt) = \tfrac{1}{2}\left[\sin(m+n)t + \sin(-m+n)t\right]$$

A deep theorem says that for any $f \in C(-\pi,\pi)$ (actually only needs to be piecewise cont.), and for the projection formula

$$f = \operatorname{proj}_{V_n} f + \underset{\substack{\cap \\ V_n^{\perp}}}{f_n^{\perp}}, \quad \text{then} \quad \lim_{n\to\infty} \| f_n^{\perp} \| = 0,$$

so that $\|f\|^2 = \lim_{n\to\infty} \|\operatorname{proj}_{V_n} f\|^2$

<u>example</u> $f(t) = t \sim 2\left[\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 5t}{5} + \dots\right]$

i.e. $\operatorname{proj}_{V_n} f = 2 \sum_{j=1}^{n} (-1)^j \frac{\sin jt}{j}$

by Pythagorean thm: $\|\operatorname{proj}_{V_n} f\|^2 = 4 \sum_{j=1}^{n} \frac{1}{j^2} \xrightarrow{\lim_{n\to\infty}} 4\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\right)$

$\lim_{n\to\infty}$, by thm

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2}{3}\pi^2$$

$$\text{SO: } \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{1}{6}\pi^2$$

MAGIC!