

Complex eigenvalues/eigenvectors example
Math 2270-3
Wednesday November 25, 2009

Glucose–insulin model (see discussion starting on page 343 of the text.)

Let $G(t)$ be the *excess* glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t . When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting the hormone insulin in order to metabolise the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{aligned}G(t + 1) &= a G(t) - b H(t) \\H(t + 1) &= c G(t) + d H(t)\end{aligned}$$

Explain (understand) the signs of the matrix coefficients. In addition to a and d being positive, why would you expect them to be less than one?

A particular choice of constants could lead to a linearized model for you! For example

$$\begin{bmatrix} G(t + 1) \\ H(t + 1) \end{bmatrix} = \begin{bmatrix} 0.9 & -0.4 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} G(t) \\ H(t) \end{bmatrix}$$

Now let's solve the initial value problem, say right after a big Thanksgiving meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

Since this is a discrete dynamical system with constant transition matrix, we know that for

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> with(LinearAlgebra) :
  A := Matrix(2, 2, [.9, -.4, .1, .9]);
                                     A :=  $\begin{bmatrix} 0.9 & -0.4 \\ 0.1 & 0.9 \end{bmatrix}$ 

```

(1)

the solution at time t is just given by A^t times the initial vector. So we can make a table of values, but it won't help us understand the big picture:

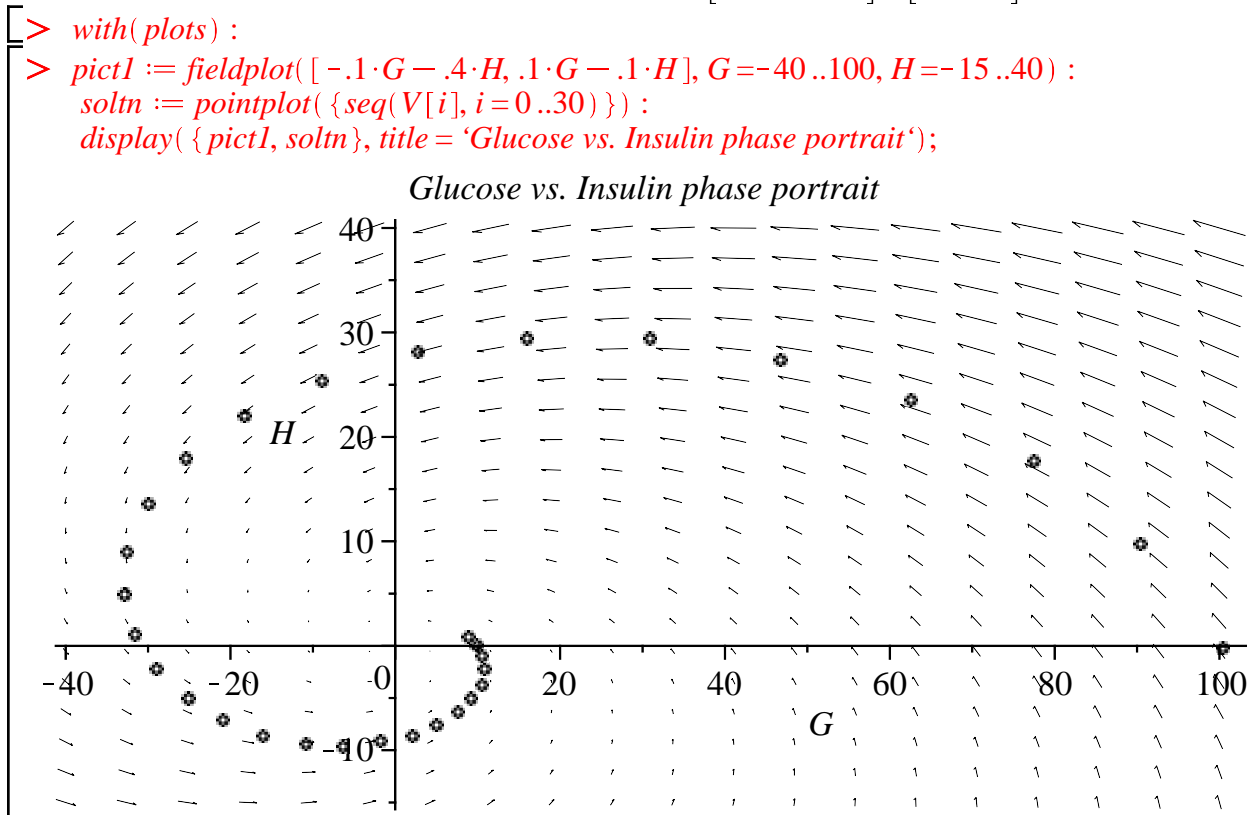
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> Digits := 5 :
> G0 := 100.0 : H0 := 0.0 : # initially G=100, H=0.
  V[0] := Vector([G0, H0]) :
  print(0, V[0][1], V[0][2]); # time t=0, initial data
  for i from 1 to 30 do
    V[i] := A.V[i - 1] :
    print(i, evalf(V[i][1]), evalf(V[i][2]));
  end do:

```

0, 100.0, 0.
1, 90., 10.
2, 77., 18.
3, 62.100, 23.900
4, 46.330, 27.720
5, 30.609, 29.581
6, 15.716, 29.684
7, 2.2706, 28.287
8, -9.2712, 25.685
9, -18.618, 22.190
10, -25.632, 18.109
11, -30.313, 13.735
12, -32.775, 9.3300
13, -33.230, 5.1195
14, -31.955, 1.2846
15, -29.273, -2.0393
16, -25.530, -4.7627
17, -21.072, -6.8394
18, -16.229, -8.2627
19, -11.301, -9.0593
20, -6.5471, -9.2835
21, -2.1790, -9.0098
22, 1.6428, -8.3267
23, 4.8092, -7.3298
24, 7.2602, -6.1159
25, 8.9805, -4.7783
26, 9.9938, -3.4024
27, 10.355, -2.0628
28, 10.145, -0.82096
29, 9.4588, 0.27563
30, 8.4027, 1.1940

We get a better geometric understanding if we plot the points onto the G–H plane, along with the deviation vector field, which points in the direction of $\begin{bmatrix} G(t+1) \\ H(t+1) \end{bmatrix} - \begin{bmatrix} G(t) \\ H(t) \end{bmatrix}$:



That’s a beautiful spiraling picture. Can you explain what’s going on qualitatively, in terms of glucose and insulin?

For Coyotes and roadrunners we were able to find a formula for the solution using eigenvector analysis. Can we do the same thing here? It turns out the answer is yes, but we’ll need to use complex eigenvalues and eigenvectors, and review complex number algebra and geometry first. Here’s the reason:

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> A := Matrix(2, 2, [ 9/10, -4/10, 1/10, 9/10 ]) : # this way Maple will do exact arithmetic
  Eigenvectors(A);

```

$$\begin{bmatrix} \frac{9}{10} + \frac{1}{5}I \\ \frac{9}{10} - \frac{1}{5}I \end{bmatrix}, \begin{bmatrix} 2I & -2I \\ 1 & 1 \end{bmatrix} \quad (3)$$

After we work with complex numbers enough, we'll realize the closed form solution for $G(t)$, and $H(t)$ in this problem is given by

$$\text{> theta} := \arctan\left(\frac{2}{9}\right);$$

$$G := t \rightarrow 25 \cdot (.85)^{\frac{t}{2}} \cdot 4 \cdot \cos(\text{theta} \cdot t);$$

$$H := t \rightarrow 25 \cdot (.85)^{\frac{t}{2}} \cdot 2 \cdot \sin(\text{theta} \cdot t);$$

$$G := t \rightarrow 100 \cdot 0.85^{\frac{1}{2}t} \cos(\theta t)$$

$$H := t \rightarrow 50 \cdot 0.85^{\frac{1}{2}t} \sin(\theta t)$$

(4)

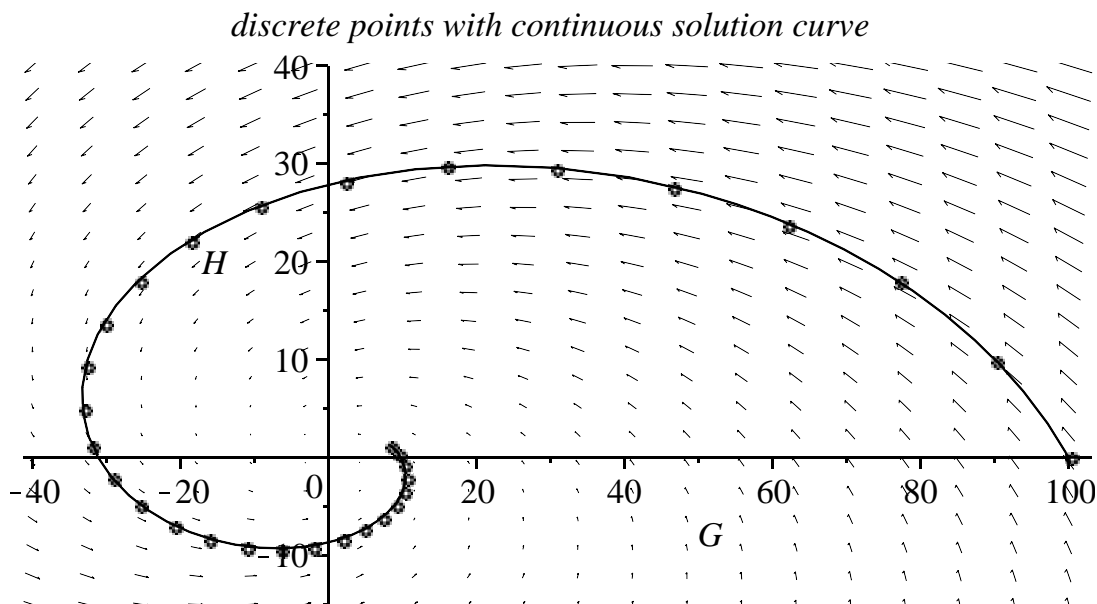
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Explain why the parametric curve with points given by $(G(t), H(t))$ describes a counterclockwise spiral converging to the origin.

Here's a picture that verifies your reasoning:

> `plot3 := plot([G(t), H(t), t = 0..30], G = -40..100, H = -15..40, color = black) :`

> `display({ pict1, soltn, plot3}, title = 'discrete points with continuous solution curve');`



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