Math 2270-3
Wed. Nov 25

HW for Friday Dec. 4
7.5 1, 2 3 4 5 6, 9, 11, 21, 24 30, 31, 32 41 45, 47
7.6 1, 3, 4, 11, 12, 17, 20 37
Chapter 7 review T/F, with justification, multiples of 6

- Tuesday Thms, p. 2!

Then begin §7.5: complex eigenvalues and eigenvectors.

Warmup: from Monday notes (discussed) yesterday we know that

$$\Theta = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{is a rotation.}$$

- Find the axis. then look at the complex eigenvalues & guess the rotation Θ.
Be clever to check answer.

Then discuss Maple notes on glucose-insulin model.

This should motivate you to start looking at §7.5 and the notes after page 1 today, about complex number algebra and geometry.

Complex number algebra and geometry

$\mathbb{C} := \{a+bi \text{ s.t. } a,b\in\mathbb{R}\}$

If $\begin{matrix}\vec{z} = a+bi \\ \vec{w} = c+di\end{matrix}$ then $\vec{z}=\vec{w}$ iff $a=c,\ b=d$.

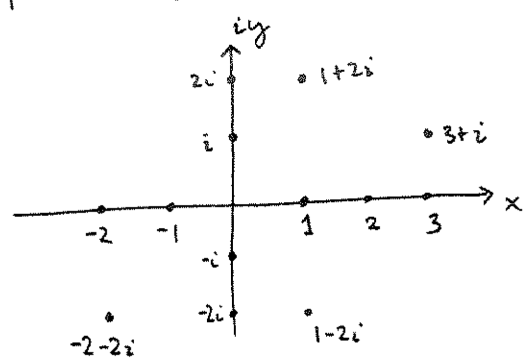
and $\vec{z}+\vec{w} := (a+c) + (b+d)i$

$\mathbb{C}$ is a real vector space (i.e. with real number scalars), of dimension 2.

With respect to the natural basis $\mathcal{B} = \{1, i\}$ the coordinate map gives (the usual) isomorphism to $\mathbb{R}^2$:

$$\vec{z} = a+bi, \quad [\vec{z}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

If we identify complex numbers with these coordinates, we get the "complex plane" representation of $\mathbb{C}$



Interesting geometry starts happening when you combine the geometry of the complex plane with algebraic operations such as complex multiplication and conjugation (complex addition $\leftrightarrow$ $\mathbb{R}^2$ vector addition
real mult $\leftrightarrow$ $\mathbb{R}^2$ scalar mult.)

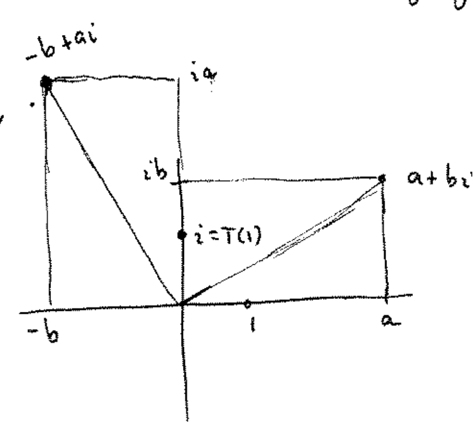
<u>Def</u> For $\begin{matrix} z = a+bi \\ w = c+di \end{matrix}$ $\quad zw = (a+bi)(c+di) := (ac-bd) + i(bc+ad)$ $\quad$ [this is equivalent to saying $i^2=-1$]

check: $zw = wz$

<u>Example</u> Let $T(z) := iz$. Describe T geometrically

$$T(a+bi) = i(a+bi)$$
$$= -b+ai$$

<u>Description</u>: Include its matrix with respect to $\{1, i\}$.

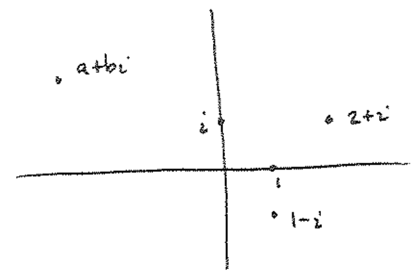


Another operation on complex numbers is <u>conjugation</u>

<u>Def</u> let $z = a + bi \in \mathbb{C}$

$\bar{z} := a - bi$

Describe conjugation geometrically:



<u>Def</u> let $z = a+bi$

$|z|^2 = a^2 + b^2$ (We call $|z|$ the <u>modulus</u> of z; it's just the magnitude of $\begin{bmatrix} a \\ b \end{bmatrix}$)

$= z\bar{z}$ (check)

also check

$\overline{zw} = \bar{z}\,\bar{w}$

$zw = 0$ iff $z = 0$ or $w = 0$

if $z \neq 0$, $\frac{1}{z}$ exists (i.e. a multiplicative inverse); in fact $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Polar form of complex numbers ⟷ corresponds to polar coords in $\mathbb{R}^2$ for $\begin{bmatrix} a \\ b \end{bmatrix}$.

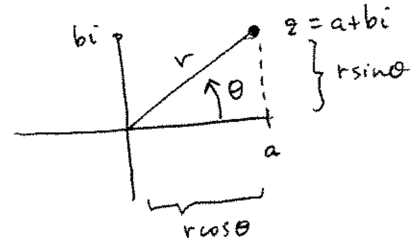
Let $z = a+bi$

Let $r = \sqrt{a^2+b^2} = |z|$ (r is polar coord radius)

Then $z = r\left(\frac{a}{\sqrt{a^2+b^2}} + i\frac{b}{\sqrt{a^2+b^2}}\right)$

$= r(\cos\theta + i\sin\theta)$

(θ is polar coord angle).



<u>Multiplication!</u> if $z = a+bi = r(\cos\theta + i\sin\theta)$

$w = c+di = \rho(\cos\phi + i\sin\phi)$

then $zw = r\rho(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

$= r\rho\left(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\right)$

$= r\rho\left(\underbrace{\cos(\theta+\phi) + i\sin(\theta+\phi)}_{\text{unit modulus.}}\right)$

$\boxed{\text{When you multiply complex numbers, the moduli multiply and the polar angles add!}}$

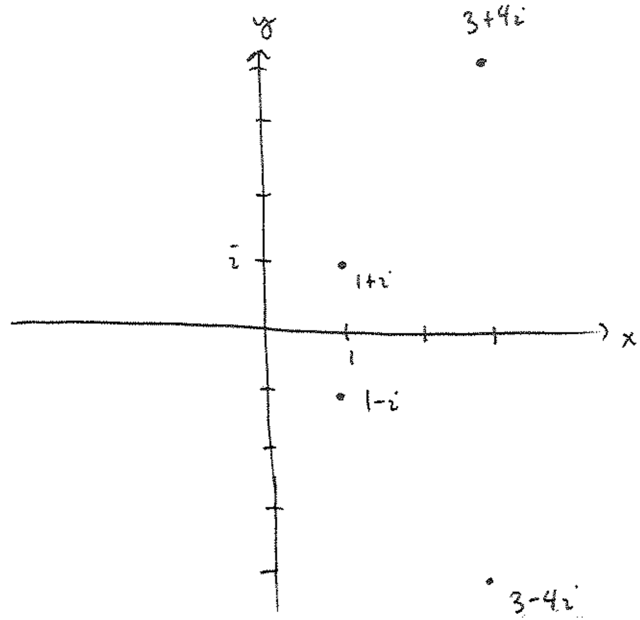
play with complex multiplication $\underbrace{\text{algebraically}}_{\text{rectangular coord form}}$ and $\underbrace{\text{geometrically}}_{\text{polar form}}$

$(3-4i)(3+4i)$

$(1+i)^2$

$(1+i)^3$

$(1-i)^4$



Ties in to <u>Euler's formula</u> (from Taylor series in Calc?)

$$\boxed{e^{i\theta} := \cos\theta + i\sin\theta}$$

rationale:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \ldots$$

$\cos\theta$! $\quad i\sin\theta$!

using this definition, we rewrite z, w and multiplication property from previous page

$$\begin{aligned} z &= re^{i\theta} \\ w &= \rho e^{i\phi} \end{aligned} \implies zw = r\rho\, e^{i(\theta+\phi)}$$

just what you'd expect if you guessed that rules of exponents hold!

<u>Example</u> let $T(z) := (1+i)z$, describe geometrically:

① Using polar form

$$1+i = \sqrt{2}\left(\tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}}\right) = \sqrt{2}\, e^{i\pi/4}$$

So $T(re^{i\theta}) = \sqrt{2}\, e^{i\pi/4}\, re^{i\theta} = \sqrt{2}\, r\, e^{i(\theta + \pi/4)}$

So T dilates by $\sqrt{2}$, and rotates by $\pi/4$

② Using real coords:

$$T(a+bi) = (1+i)(a+bi) = (a-b) + i(a+b)$$

$$[T]_{\{1,i\}} = A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \sqrt{2}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

rotation dilation matrix! agrees with ①

To be continued...