Math 2270-3
Tuesday Nov. 24 7.2-7.3 odds and ends

- HW questions ~ 20 minutes ~ this may lead to p. 2-3 Monday (3×3 orthog matrices) and/or to matrix powers for diagonal matrices (page 1 today)

- Example page 1 Monday to motivate today's theorem (page 2 today)

- Matrix power magic!

If $B = S^{-1}AS$ then $B^2 = S^{-1}A\underbrace{SS^{-1}}_{I}AS = S^{-1}A^2S$

$B^3 = S^{-1}ASS^{-1}A^2S = S^{-1}A^3S$

so, inductively, $B^t = S^{-1}A^tS$ $t = 1, 2, \ldots$

i.e. A, B similar $\Rightarrow A^t, B^t$ are too! $t \in \mathbb{N}$.
(useful in one of your hw's to show to matrices aren't similar.)

If $D = S^{-1}AS$, D diagonal, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \Rightarrow D^2 = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ 0 & & \ddots \; \lambda_n^2 \end{bmatrix}$

$\Rightarrow D^t = \begin{bmatrix} \lambda_1^t & & 0 \\ & \lambda_2^t & \\ 0 & & \ddots \; \lambda_n^t \end{bmatrix}$

so $A = SDS^{-1}$

$A^t = \underbrace{SD^tS^{-1}}_{\text{efficient to compute.}}$

<u>Example</u>: Compute A^t for $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

ans $\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4^t & 0 \\ 0 & (-2)^t \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

<u>Theorem</u> Let $f_A(\lambda) = |A - \lambda I| = (-1)^n (\lambda-\lambda_1)^{k_1} (\lambda-\lambda_2)^{k_2} \cdots (\lambda-\lambda_\ell)^{k_\ell}$ $\quad k_1+k_2+\cdots+k_\ell = n$, λ_i's distinct

Then

- $\dim(E_{\lambda_i}(A)) \le k_i \quad i=1,2,\ldots \ell$ (geom. mult $\le$ alg. mult.)
- A is diagonalizable if and only if each $\dim(E_{\lambda_i}(A)) = k_i \quad i=1,2,\ldots \ell$
  In this case an eigenbasis for A may be constructed by amalgamating bases for each $E_{\lambda_i}(A)$; these n vectors will be lin ind.!

<u>proof</u>: • $\dim(E_{\lambda_i}(A)) \le k_i$:

Let $\{\vec v_1, \vec v_2, \ldots \vec v_s\}$ a basis for $E_{\lambda_i}(A)$, <u>extend</u> to a basis $\mathcal{B} = \{\vec v_1, \vec v_2, \ldots \vec v_s, \vec v_{s+1}, \ldots \vec v_n\}$ for $\mathbb{R}^n$

For $T(\vec x) = A\vec x$, let $B = [T]_{\mathcal{B}}$.

Then

$$B = \begin{array}{c} \\ s\left\{\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right. \\ \\ \end{array} \left[\begin{array}{c|c} \overbrace{\begin{matrix}\lambda_i & & O\\ & \lambda_i & \\ O & & \lambda_i\end{matrix}}^{s} & E \\ \hline O & F \end{array}\right]$$

$$\underbrace{\qquad}_{s} \qquad \underbrace{\qquad}_{n-s}$$

$\Rightarrow f_B(\lambda) = (\lambda - \lambda_i)^s \, |F - \lambda I|$

But A and B are similar, so have same characteristic polynomial. Thus $s \le k_i$ ∎

• A diagonalizable iff each $\dim(E_{\lambda_i}(A)) = k_i$

$\Rightarrow$ let A be diagonalizable, i.e. $AS = SD$

S: cols are eigenbasis of $\mathbb{R}^n$; D: diagonal

Then since D is similar to A eigenvalue λ_i appears k_i times along D's diagonal $\Rightarrow$ S contains k_i eigenvectors with eval λ_i $\Rightarrow \dim(E_{\lambda_i}(A)) \ge k_i$. Since $\le$ holds by 1st bullet, deduce $\dim = k_i$ ∎

$\Leftarrow$ If each $\dim(E_{\lambda_i}(A)) = k_i$, find bases for each eigenspace, and group them into a single collection of n vectors:

$$\Big\{ \underbrace{\vec v_1, \ldots \vec v_{k_1}}_{E_{\lambda_1}\text{ basis}},\ \underbrace{\vec v_{k_1+1}, \ldots \vec v_{k_1+k_2}}_{E_{\lambda_2}\text{ basis}},\ \ldots,\ \underbrace{\vec v_{n-k_\ell+1}, \ldots \vec v_n}_{E_{\lambda_\ell}\text{ basis}} \Big\}$$

Check $\mathbb{R}^n$ basis by checking lin. ind.

$$\text{Let } \underbrace{(c_1\vec v_1 + \cdots + c_{k_1}\vec v_{k_1})}_{\vec w_1 \in E_{\lambda_1}(A)} + \underbrace{(c_{k_1+1}\vec v_{k_1+1} + \cdots + c_{k_1+k_2}\vec v_{k_1+k_2})}_{\vec w_2 \in E_{\lambda_2}(A)} + \cdots \underbrace{+ c_n \vec v_n}_{\vec w_\ell \in E_{\lambda_\ell}(A)} = \vec 0$$

$$\vec w_1 + \vec w_2 + \cdots \vec w_\ell = \vec 0$$

Since ~~non-zero~~ eigenvectors with different eigenvalues are linearly independent, deduce $\vec w_1 = \vec w_2 = \ldots = \vec w_\ell = \vec 0$!

$\vec w_1 = \vec 0 \Rightarrow c_1\vec v_1 + \cdots + c_{k_1}\vec v_{k_1} = 0 \Rightarrow c_1 = c_2 = \ldots c_{k_1} = 0$; same for all $\vec w_j = \vec 0$

$\Rightarrow c_1 = c_2 = \ldots = c_n = 0$ ∎