Math 2270-3
Monday Nov. 23

- Review eigenvalue / eigenvector concepts.
- prove linear independence theorem & corollary, p. 5 Wed notes
- pages 3-4 Friday about the algebra & geometry of similar matrices
  play the "are these matrices similar?" game p. 4.

Example Find bases for each eigenspace, determine algebraic & geometric multiplicities

$$A = \begin{bmatrix} \begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix} & & 0 \\ & \begin{matrix} 3 & 1 \\ 0 & 3 \end{matrix} & \\ 0 & & \begin{matrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{matrix} \end{bmatrix}$$

Compare to
the theorem below (proof tomorrow)

Theorem Let $A_{n\times n}$ have eigenvalue λ_j with algebraic multiplicity k_j.
Then $\dim(E_{\lambda_j}(A)) \le k_j$

Theorem Suppose $A_{n\times n}$ has characteristic polynomial $f_A(\lambda)$ that factors entirely over the real numbers, $f_A(\lambda) = (-1)^n(\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_\ell)^{k_\ell}$ $\quad k_1+k_2+\cdots+k_\ell = n$ $\quad \lambda_1, \ldots \lambda_\ell$ distinct & real
Then A is diagonalizable
iff each $\dim(E_{\lambda_j}(A)) = k_j$

<u>Example</u> : We've checked that the only 2×2 orthogonal matrices Q are rotations and reflections, depending on whether $|Q| = +1$ or $|Q| = -1$.

In $\mathbb{R}^n$ we <u>call</u> an orthogonal matrix Q ($Q^TQ = I$) a rotation matrix when $|Q| = +1$; in $\mathbb{R}^3$ this actually means its a geometric rotation with a single axis:

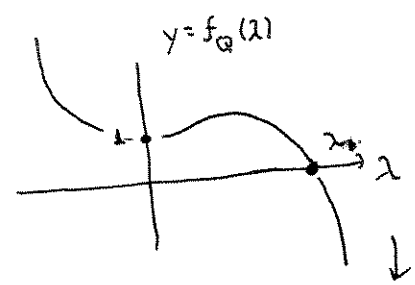
<u>Theorem</u> : Let $Q_{3\times 3}$ be orthogonal with $\det(Q) = +1$. Then unless $Q = I$, there is a unique axis $\text{span}\{\vec{u}\} = L$ s.t. Q is rotation about this axis, by some angle θ.

<u>proof</u> : Let Q orthog, $|Q| = 1$ ($|Q| = \pm 1$ for any orthog matrix since $Q^TQ = I$).

Then

$$f_Q(\lambda) = -\lambda^3 + (\text{trace}\, Q)\lambda^2 + c_1\lambda + \underset{\substack{\uparrow \\ |Q|.}}{1}$$

$$f_Q(0) = 1, \quad \lim_{\lambda\to\infty} f_Q(\lambda) = -\infty$$



so $\exists$ a positive eigenvalue λ_1:

(intermediate value thm!).

Let $\vec{u}$ be a unit eigenvector with eigenvalue $\lambda_1 > 0$

$$Q\vec{u} = \lambda_1\vec{u}$$

Q preserves length so $\|\vec{u}\| = 1 = \|Q\vec{u}\| = \lambda_1\|\vec{u}\| \Rightarrow \lambda_1 = 1$

so $Q\vec{u} = \vec{u}$.

Let $L = \text{span}\{\vec{u}\}$. This will be our axis.

complete to an orthornormal basis of $\mathbb{R}^3$, $\mathcal{B} = \{\vec{u}, \underbrace{\vec{v}, \vec{w}}_{\text{basis of } L^\perp}\}$

positively oriented

$$B = [Q]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{bmatrix}$$

$[Q\vec{u}]_{\mathcal{B}}$

$[Q\vec{v}]_{\mathcal{B}}$: $\vec{u}\cdot\vec{v} = 0$
$\Rightarrow Q\vec{u}\cdot Q\vec{v} = 0$
$\Rightarrow \vec{u}\cdot Q\vec{v} = 0$

$[Q\vec{w}]_{\mathcal{B}}$: $\vec{u}\cdot\vec{w} = 0$
$\Rightarrow Q\vec{u}\cdot Q\vec{w} = 0$

$Q\vec{v} = a\vec{v} + b\vec{w}$
$Q\vec{w} = c\vec{v} + d\vec{w}$

$\|Q\vec{v}\|^2 = 1 \Rightarrow a^2 + b^2 = 1$
$\|Q\vec{w}\|^2 = 1 \Rightarrow c^2 + d^2 = 1$.

$Q\vec{v}\cdot Q\vec{w} = 0 \Rightarrow ac + bd = 0$.

so $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$ or $\begin{bmatrix} b \\ -a \end{bmatrix}$.

$|B| = 1$ since B similar to Q

$$\Rightarrow B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad \blacksquare$$

Note: Q is not diagonalizable over $\mathbb{R}$, but it is if we consider the <u>complex</u> vector space $\mathbb{C}^3$!!

If $Q_{3\times3}$ is orthogonal, $|Q|=-1$, what is its geometric action?

<u>Example</u> : For rotation $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ find axis vector & rotation!