Math 2270-3
Friday Nov. 20
§7.2-7.4

HW for Wed. Nov. 25 (1)

7.3 7, 13, 20, 23, 27, 28, 35, 36, 38, 44
7.4 14, 17, 23, 25, 27, 31, 35, 37, 38, 47, 54

Def'ns and algorithms:

$A_{n\times n}$

- $\vec{v}$ is an <u>eigenvector</u> of A with <u>eigenvalue</u> λ means:

- <u>Characteristic polynomial</u> $f_A(\lambda) := \det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{vmatrix}$

$$* \; f_A(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \underbrace{(\text{trace} A)}_{\sum a_{ii}} \lambda^{n-1} + c_{n-2}\lambda^{n-2} + \dots + c_1 \lambda + \det A \qquad \text{(by pattern expansion)}$$

- the roots λ_j of $f_A(\lambda)$, i.e. $f_A(\lambda_j)=0$ are exactly the eigenvalues of A, since
$$f_A(\lambda_j) = 0 \quad \text{iff} \quad \det(A-\lambda_j I) = 0$$
$$\text{iff} \quad \dim(\ker(A-\lambda_j I)) \geq 1.$$

- the λ_j-eigenspace $E_{\lambda_j}(A) := \ker(A-\lambda_j I) = \{\vec{v} \in \mathbb{R}^n \text{ s.t. } A\vec{v} = \lambda_j \vec{v}\}$
  - $\dim(E_{\lambda_j}(A)) :=$ <u>geometric multiplicity</u> of λ_j

By the fundamental theorem of algebra, $f_A(\lambda)$ factors completely over $\mathbb{C}$, so

$$* \; f_A(\lambda) = (-1)^n (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) \qquad \lambda_j \in \mathbb{C}, \; 1 \leq j \leq n$$

Comparing λ^{n-1} coeffs in formulas, deduce $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n a_{ii} = \text{trace}(A)$

" 1 " " " $(\lambda_1 \lambda_2 \cdots \lambda_n) = \det(A)$

Collecting identical factors, ~~we~~ (and renaming), we may write

$$* \; f_A(\lambda) = (-1)^n (\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_\ell)^{k_\ell} \qquad \lambda_1, \lambda_2, \dots \lambda_\ell \text{ distinct}, \; k_1+k_2+\dots+k_\ell = n$$

- If $(\lambda-\lambda_j)^{k_j}$ is such a factor, k_j is called the <u>algebraic multiplicity of λ_j</u>.

- Matrices for which there is an $\mathbb{R}^n$-basis of eigenvectors are special:

If $A_{n\times n}$ has an $\mathbb{R}^n$ basis of eigenvectors, $\mathcal{B}=\{\vec{v}_1,\vec{v}_2,..\vec{v}_n\}$ with $A\vec{v}_j=\lambda_j\vec{v}_j$ $j=1,2,...n$
We say that <u>A is diagonalizable</u>

This is equivalent to there being an $\mathbb{R}^n$ basis $\mathcal{B}=\{\vec{v}_1,\vec{v}_2,..\vec{v}_n\}$ so that for $T(\vec{x}):=A\vec{x}$
the matrix $[T]_{\mathcal{B}}=D:=\begin{bmatrix}\lambda_1 & & O\\ & \lambda_2 & \\ O & & \ddots\ \lambda_n\end{bmatrix}$ is diagonal, (i.e the linear transformation $T(\vec{x})=A\vec{x}$ is diagonalizable would've been better language)

Or equivalently, A is similar to some diagonal matrix D

Even if you're shaky on change of basis it's possible to keep the diagonal matrix and similarity transformations straight if you can do matrix algebra!

$$A\left[\vec{v}_1\middle|\vec{v}_2\middle|\cdots\middle|\vec{v}_n\right]=\left[\lambda_1\vec{v}_1\middle|\lambda_2\vec{v}_2\middle|\cdots\lambda_n\vec{v}_n\right]$$

$\{\vec{v}_1,\vec{v}_2,\ldots\vec{v}_n\}$ eigenvectors of A, evals $\lambda_1,\lambda_2\ldots\lambda_n$

$$=\left[\vec{v}_1\middle|\vec{v}_2\middle|\cdots\vec{v}_n\right]\begin{bmatrix}\lambda_1 & 0 & 0 & & 0\\ 0 & \lambda_2 & 0 & & 0\\ 0 & 0 & \lambda_3 & & 0\\ \vdots & & 0 & \ddots & \lambda_n\end{bmatrix}$$

rewrite

$$AS=SD$$

S^{-1} exists iff $\{\vec{v}_1,\vec{v}_2,\ldots\vec{v}_n\}$ are $\mathbb{R}^n$ basis

$$S^{-1}AS=D$$

$$S=S_{E\leftarrow\mathcal{B}}=\left[\vec{v}_1\middle|\vec{v}_2\middle|\cdots\vec{v}_n\right].$$

- Do example on p.4 Wed.
  Include algebraic & geometric multiplicity for each eigenvalue
  Check $AS=SD$ as above

- Prove the important linear independence theorem, page 5 Wednesday, and its corollary (which gives a shortcut for page 4 example.)

- Do example p. 6 Wed $\leadsto$ notice contrast in algebraic vs. geometric multiplicity.

Eigenspace analysis can (help) determine whether or not matrices are similar.

<u>Theorem</u> Let A and B be similar matrices,

$$B = S^{-1}AS \qquad (A = SBS^{-1})$$

Then

a) $f_A(\lambda) = f_B(\lambda)$, i.e. their characteristic polys are the same.

Thus : • <u>same eigenvalues</u>, <u>same algebraic multiplicity</u>.
• <u>same trace</u>, <u>same determinant</u>

proof :
$$|B - \lambda I| = |S^{-1}AS - \lambda I|$$
$$= |S^{-1}AS - \lambda S^{-1}S|$$
$$= |S^{-1}(AS - \lambda S)|$$
$$= |S^{-1}(A - \lambda I)S| = \cancel{|S^{-1}|}\,|A - \lambda I|\,\cancel{|S|} \quad \blacksquare$$

b) Consider the $\mathbb{R}^n$ isomorphism
$$T(\vec{v}) := S\vec{v}$$
Then $T: E_{\lambda_i}(B) \to E_{\lambda_i}(A)$ is an isomorphism between the eigenspaces.

Thus • <u>same geometric multiplicities</u> (because isomorphic spaces have same dimension!)

<u>proof</u> : $SB = AS$.

If $B\vec{v} = \lambda\vec{v}$
then $A(S\vec{v}) = SB\vec{v} = S\lambda\vec{v} = \lambda S\vec{v}$.
Conversely, if $A\vec{w} = \lambda\vec{w}$
then
$$B(S^{-1}\vec{w}) = S^{-1}ASS^{-1}\vec{w}$$
$$= S^{-1}A\vec{w}$$
$$= S^{-1}\lambda\vec{w} = \lambda(S^{-1}\vec{w})$$
$\blacksquare$

Play the similarity game!

$E_5(A) = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$
$E_{-1}(A) = \text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$

We showed Wed. that $A = \begin{bmatrix}1 & 2\\4 & 3\end{bmatrix}$ is similar to $\begin{bmatrix}-1 & 0\\0 & 5\end{bmatrix}$

- Is A similar to $\begin{bmatrix}-1 & 0\\0 & 3\end{bmatrix}$?

- Is A similar to $\begin{bmatrix}2 & 3\\3 & 2\end{bmatrix}$?

- Is A similar to $\begin{bmatrix}2 & 1\\7 & 1\end{bmatrix}$?

from p.6 Wed:

- Is $A = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}$ similar to $C = \begin{bmatrix}2 & 1 & 0\\0 & 2 & 1\\0 & 0 & 2\end{bmatrix}$?

new example:

- Is $A = \begin{bmatrix}1 & 0 & 1\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$ similar to $B = \begin{bmatrix}3 & 0 & 0\\1 & 2 & 0\\7 & 8 & 1\end{bmatrix}$? Hint: be clever!

By the way, in general, even if A and B pass all the tests on page 3 they may fail to be similar!!