Math 2270-3
Wednesday Nov. 18

§7.2-7.3: Eigenvalues and eigenvectors

Recall from the coyote-road runner discrete dynamical system example the importance of:

- If $A_{n \times n}$ and $\vec{v} \neq \vec{0}$ with $A\vec{v} = \lambda \vec{v}$
  then $\vec{v}$ is an <u>eigenvector</u> of A with <u>eigenvalue λ</u>

How to find eigenvalues, then eigenvectors:

$$A\vec{v} = \lambda\vec{v}$$
iff $A\vec{v} - \lambda\vec{v} = \vec{0}$
iff $A\vec{v} - \lambda I \vec{v} = \vec{0}$
iff $(A - \lambda I)\vec{v} = \vec{0}$. For $\vec{v} \neq \vec{0}$ this can happen iff $A - \lambda I$ is <u>not</u> invertible
i.e. $\det(A - \lambda I) = 0$

<u>step 1</u>: Compute $\det(A - \lambda I) = \det \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn}-\lambda \end{bmatrix}$

$\quad\quad\quad\quad\quad /\!/$

$\quad\quad\quad\quad\quad p(\lambda)$

$\boxed{\text{called the characteristic polynomial}}$

$\left.\begin{array}{l}\text{this is a polynomial in } \lambda \text{ of degree } n, \\ p(\lambda) = (-1)^n \lambda^n + \ldots + \det A\end{array}\right\}$ Why?!! (use pattern analysis!!)

Its roots are eigenvalues.

<u>step 2</u>: For each eigenvalue λ the
$\{\vec{v} \mid (A - \lambda I)\vec{v} = 0\}$ is the kernel of $A - \lambda I$
so is a subspace, and has a basis of eigenvectors, which we find in the usual way by computing $\text{rref}(A - \lambda I \,\vdots\, \vec{0})$ and backsolving.

<u>step 3</u>: In applications we often hope there is a basis of $\mathbb{R}^n$ made out of eigenvectors of A (like in the Coyote-roadrunner problem). This will not always be the case, although under certain conditions on A such a result is guaranteed.

Example : $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

- Find the eigenvalues λ of A

- For each eigenvalue find an eigenbasis for the corresponding eigenspace.

- Is there a basis of $\mathbb{R}^2$ made out of eigenvectors of A?
  What is the matrix of $f(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\vec{x}$ with respect to this basis?
  Find this matrix B two ways!

Example (from yesterday)

$A = \begin{bmatrix} .86 & .08 \\ -.12 & 1.14 \end{bmatrix}$ eigenvalues & eigenvectors?

2×2 case : $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A-\lambda I) &= (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \\ &= \lambda^2 - \underset{\substack{\uparrow \\ \text{sum of diagonal elts}}}{\mathrm{trace}(A)}\,\lambda + \det A \end{aligned}$$

(also for n×n)

e.g. page 2:
$\lambda^2 - 4\lambda - 5$

3×3 case : $A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$

Can you show

$$\det(A-\lambda I) = -\lambda^3 + (\mathrm{trace}(A))\lambda^2 - (\underset{\downarrow}{\det(A_{1,1})} + \underset{\downarrow}{\det(A_{2,2})} + \underset{\downarrow}{\det(A_{3,3})})\lambda + \det A \quad ?$$

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Any guesses on how this generalizes to the n×n case?

Example

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

- What are the eigenvalues of A?

- Find eigenbases for each eigenvalue

- Is A similar to a diagonal matrix? (See thm on next page to save work in checking whether you get an $\mathbb{R}^3$ basis of A-eigenvectors).

Theorem: Let $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ be eigenvectors of $A_{n\times n}$, with distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_k$.

no two the same.

Then $\{\vec{v}_1, \vec{v}_2, - \vec{v}_k\}$ are linearly independent.

proof: Let

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$$

apply A: $\Rightarrow$ $c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_k\lambda_k\vec{v}_k = \vec{0}$

A^2 $\quad c_1\lambda_1^2\vec{v}_1 + c_2\lambda_2^2\vec{v}_2 + \cdots + c_k\lambda_k^2\vec{v}_k = \vec{0}$

A^{k-1} $\quad c_1\lambda_1^{k-1}\vec{v}_1 + \cdots + c_k\lambda_k^{k-1}\vec{v}_k = \vec{0}$

in matrix form:

$$\left[\begin{array}{c|c|c|c} c_1\vec{v}_1 & c_2\vec{v}_2 & \cdots & c_k\vec{v}_k \end{array}\right]\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_k & \lambda_k^2 & & \lambda_k^{k-1} \end{bmatrix} = [\,0\,]$$

transpose of Vandermonde matrix, its

$$\det = \prod_{i>j}(\lambda_i - \lambda_j) \neq 0$$

so is invertible!
Multiply eqtn by its inverse (on the right!)

$$\Rightarrow \left[\begin{array}{c|c|c|c} c_1\vec{v}_1 & c_2\vec{v}_2 & \cdots & c_k\vec{v}_k \end{array}\right] = [\,0\,]$$

since each $\vec{v}_j \neq \vec{0}$ deduce each $c_j = 0$! ■

Corollary: If $A_{n\times n}$ has n distinct eigenvalues $\lambda_1, \lambda_2, - \lambda_n$, then there is an $\mathbb{R}^n$ basis of eigenvectors of A.

Proof: Each eigenvalue λ_i has at least one non-zero eigenvector $\vec{v}_i$. (Since the λ_i-eigenspace is at least 1-dim'l)
By theorem above, $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ are linearly independent.
since $\dim(\mathbb{R}^n) = n$ they also span $\mathbb{R}^n$ and are a basis.

If A doesn't have n distinct eigenvalues, harder to tell without working:

example $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$p(\lambda) = (2-\lambda)^3$ $p(\lambda) = (2-\lambda)^3$