Math 2270-3
Tues. Nov 17

Chapter 7: Eigenvalues and eigenvectors
(with discrete dynamical system applications)

§7.1 intro & overview example

bring along text, p. 294-299.

Coyotes and roadrunners caricature of predator prey model:

$c(t)$, $r(t)$ pops at year t

<u>Model</u>

$$c(t+1) = .86\,c(t) + .08\,r(t)$$
$$r(t+1) = -.12\,c(t) + 1.14\,r(t)$$

$$\begin{bmatrix} c(t+1) \\ r(t+1) \end{bmatrix} = \overbrace{\begin{bmatrix} .86 & .08 \\ -.12 & 1.14 \end{bmatrix}}^{A} \begin{bmatrix} c(t) \\ r(t) \end{bmatrix}$$

explain coefficients qualitatively; is this a reasonable model for what some predator-prey system might look like?

this is an example of a <u>discrete</u> <u>dynamical</u> <u>system</u>

- discrete: time changes in <u>discrete</u> units, rather than <u>continuously</u>
- dynamical: changes in time
- system: more than 1 function being studied, in general

(differences vs. derivatives)

e.g.
$$\frac{c(t+1)-c(t)}{1} \approx c'(t) = -.14\,c(t) + .08\,r(t)$$
$$\frac{r(t+1)-r(t)}{1} \approx r'(t) = -.12\,c(t) + .14\,r(t)$$
$\underbrace{\qquad\qquad\qquad}_{\text{Math 2280, e.g.}}$

Goal in discrete dynamical system: understand how the long-time behavior depends on the transition matrix A:

$$\vec{x}(0) = \begin{bmatrix} c(0) \\ r(0) \end{bmatrix} \xrightarrow{A} \vec{x}(1) \xrightarrow{A} \vec{x}(2) \xrightarrow{A} \vec{x}(3) \longrightarrow \ldots \longrightarrow \underset{A^t\vec{x}_0}{\vec{x}(t)} \longrightarrow \ldots$$

"$\vec{x}_0$"

$$\vec{x}(t) = A^t\vec{x}_0 \qquad (t = 0, 1, 2, \ldots \text{ years, e.g.})$$

e.g. for our coyotes & roadrunners: $\vec{x}_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix} \Rightarrow \vec{x}(t) = A^{10}\begin{bmatrix} 100 \\ 100 \end{bmatrix} \approx \begin{bmatrix} 80 \\ 170 \end{bmatrix}$ (technology?)

Case 1: (special choice of lots of roadrunners relative to coyotes)

$$\vec{x}_0 = \begin{bmatrix} 100 \\ 300 \end{bmatrix} \qquad \vec{x}(1) = \begin{bmatrix} .86 & .08 \\ -.12 & 1.14 \end{bmatrix}\begin{bmatrix} 100 \\ 300 \end{bmatrix} = \begin{bmatrix} 110 \\ 330 \end{bmatrix}$$

$$= (1.1)\begin{bmatrix} 100 \\ 300 \end{bmatrix} = (1.1)\vec{x}_0$$

so
$$\begin{aligned}\vec{x}(2) &= A\,\vec{x}(1) \\ &= A\,(1.1)\vec{x}_0 \\ &= (1.1)\,A\vec{x}_0 \\ &= (1.1)^2\,\vec{x}_0\end{aligned}$$

$$\vec{x}(t) = (1.1)^t\,\vec{x}_0 \;; \qquad \begin{bmatrix} c(t) \\ r(t) \end{bmatrix} = (1.1)^t\begin{bmatrix} 100 \\ 300 \end{bmatrix}$$

exponential growth both populations!

Case 2

$$\vec{x}_0 = \begin{bmatrix} c_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$; tougher times

$$\begin{array}{r} 86 \\ 2 \\ \hline 172 \end{array}$$

$$\vec{x}(1) = A\vec{x}_0 = \begin{bmatrix} .86 & .08 \\ -.12 & 1.14 \end{bmatrix}\begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 180 \\ 90 \end{bmatrix} = .9\,\vec{x}_0$$

$$\begin{aligned}\vec{x}(2) &= A(A\vec{x}_0) \\ &= A(.9\vec{x}_0) \\ &= .9\,A(\vec{x}_0) = (.9)^2\,\vec{x}_0\end{aligned}$$

$$\vec{x}(t) = .9^t\begin{bmatrix} 200 \\ 100 \end{bmatrix}$$ bad news!

Case 3

$$\vec{x}_0 = \begin{bmatrix} c_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$$

$$\vec{x}(1) = \begin{bmatrix} .86 & .08 \\ -.12 & 1.14 \end{bmatrix}\begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 940 \\ 1020 \end{bmatrix}$$

$$\vec{x}(2) = \begin{bmatrix} 890 \\ 1050 \end{bmatrix}, \quad \vec{x}(3) = \begin{bmatrix} 849.4 \\ 1090.2 \end{bmatrix}$$

$$\vec{x}(4) \doteq \begin{bmatrix} 818 \\ 1141 \end{bmatrix}, \quad \vec{x}(5) \approx \begin{bmatrix} 795 \\ 1203 \end{bmatrix}$$

...

$$\vec{x}(10) \approx \begin{bmatrix} 798 \\ 1696 \end{bmatrix}$$

... no obvious pattern.

$$\vec{x}(20) \approx \begin{bmatrix} 1443 \\ 4085 \end{bmatrix}$$

Because the standard basis of $\mathbb{R}^2$ is the WRONG basis for this problem!!

The problem is, we're using the wrong basis!

Note $\underbrace{\begin{bmatrix}1000\\1000\end{bmatrix}}_{\vec{x}_0} = 2\begin{bmatrix}100\\300\end{bmatrix} + 4\begin{bmatrix}200\\100\end{bmatrix}$

$= 2\vec{v}_1 + 4\vec{v}_2$

So $A\vec{x}_0 = A(2\vec{v}_1 + 4\vec{v}_2)$

$= 2A\vec{v}_1 + 4A\vec{v}_2$

$= 2(1.1)\vec{v}_1 + 4(.9)\vec{v}_2$

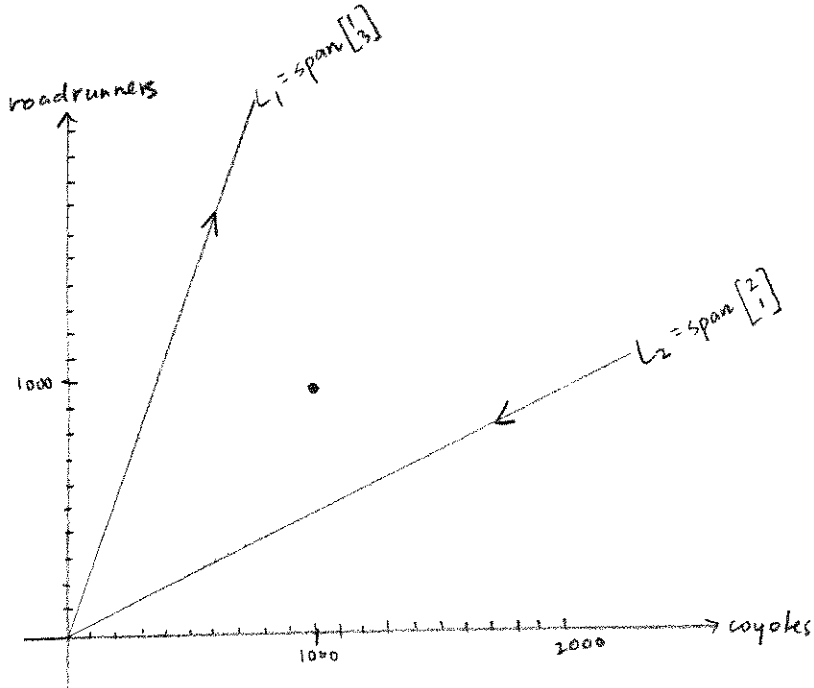
$\vec{x}(2) = A^2\vec{x}_0 = 2(1.1)^2\vec{v}_1 + 4(.9)^2\vec{v}_2$

$$\vec{x}(t) = 2(1.1)^t\begin{bmatrix}100\\300\end{bmatrix} + 4(.9)^t\begin{bmatrix}200\\100\end{bmatrix} = \begin{bmatrix}c(t)\\r(t)\end{bmatrix}$$

<u>Now</u>, you see long time behavior (into the future, and the past)

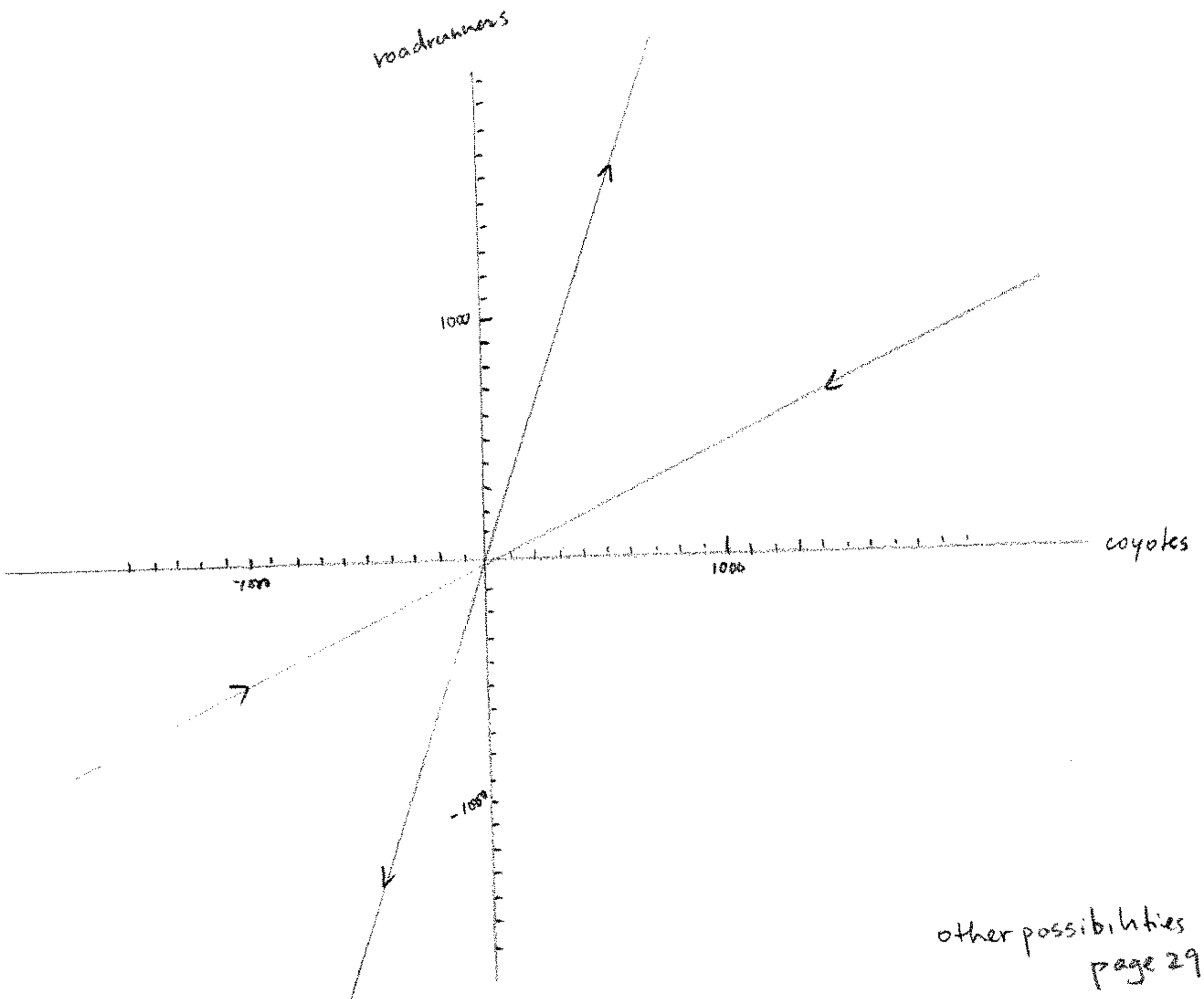
plot the $t>0$, $t<0$ behavior geometrically ~ see pages 298-299

this sort of picture is often called a "phase portrait", because these considerations first came up in physics applications



if $\begin{bmatrix} c_0 \\ r_0 \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

then $\begin{bmatrix} c(t) \\ r(t) \end{bmatrix} = d_1 (1.1)^t \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d_2 (.9)^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



other possibilities
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in real ecological systems there are non-linear terms which typically prevent total population explosion, for example

Big picture.

- $A_{n\times n}$, $\vec{v}$ is an eigenvector of A, with eigenvalue λ iff $A\vec{v} = \lambda\vec{v}$

- Application: Consider the discrete dynamical system $\begin{cases} \vec{x}(t+1) = A\vec{x}(t), & A_{n\times n} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$

  Then $\vec{x}(t) = A^t \vec{x}_0$.

  If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ consisting of eigenvectors,

  $A\vec{v}_1 = \lambda_1\vec{v}_1, \quad A\vec{v}_2 = \lambda_2\vec{v}_2, \dots \quad A\vec{v}_n = \lambda_n\vec{v}_n$

  If $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$

  Then $\vec{x}(t) = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + \dots + c_n\lambda_n^t\vec{v}_n$

This is change of basis one more time!

i.e. $$\vec{x}(t) = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \middle| \vec{v}_n\right] \begin{bmatrix} \lambda_1^t & & O \\ & \lambda_2^t & \\ O & & \ddots \; \lambda_n^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Note $S = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \dots \vec{v}_n\right] = S_{E \leftarrow \mathcal{B}}$

$S\vec{c} = \vec{x}_0$

$\vec{c} = S^{-1}\vec{x}_0$

$$x(t) = \underbrace{S}_{S_{E\leftarrow\mathcal{B}}} \underbrace{\begin{bmatrix} \lambda_1 & & O \\ O & \ddots & \lambda_n \end{bmatrix}^t}_{([T]_{\mathcal{B}})^t} \underbrace{S^{-1}}_{S_{\mathcal{B}\leftarrow E}} \vec{x}_0$$

eigenvectors and eigenvalues
for some of our favorite geometric transformations:

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (dilations)

rotations

reflections

projections

How to find
eigenvalues & eigenvectors
in general???
Chapter 7!