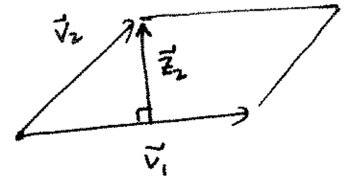
Math 2270-3
Monday Nov. 16
§6.3
Geometry of determinants: Volumes & orientation

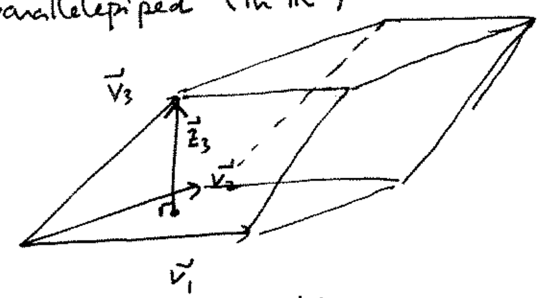
related to Gram-Schmidt ideas

parallelogram (in $\mathbb{R}^n$)



Area $= \|\vec{v}_1\| \|\vec{z}_2\|$ (base · ht)

parallelepiped (in $\mathbb{R}^n$)



Vol = (Area of base) · ht
$= \|\vec{v}_1\| \|\vec{z}_2\| \|\vec{z}_3\|$

in higher dimensions,
Vol of parallelepiped spanned by $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is defined to be $\boxed{\|\vec{v}_1\| \|\vec{z}_2\| \cdots \|\vec{z}_k\| = \text{Vol}}$
(book: $\|\vec{v}_1\| \|\vec{v}_2^{\perp}\| \cdots \|\vec{v}_k^{\perp}\|$)

Gram-Schmidt!

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{v}_2^{\perp} = \vec{z}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$
$$\vec{u}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|}$$

$$\vec{z}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$
$$\vec{u}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|}$$

$\vdots$

$$\vec{z}_k = \vec{v}_k - \sum_{j=1}^{k-1} (\vec{v}_k \cdot \vec{u}_j)\vec{u}_j$$
$$\vec{u}_k = \frac{\vec{z}_k}{\|\vec{z}_k\|}$$

QR decomp

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_k \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ & r_{22} & & \vdots \\ & 0 & \ddots & \\ & & & r_{kk} \end{bmatrix}$$

$$A = QR$$
$$Q^T A = Q^T Q R = IR = R$$

so
$$R = \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vdots \\ \vec{u}_k^T \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix}$$

$$\left.\begin{aligned} r_{11} &= \vec{u}_1 \cdot \vec{v}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} \cdot \vec{v}_1 = \frac{\|\vec{v}_1\|^2}{\|\vec{v}_1\|} = \|\vec{v}_1\|. \\ r_{22} &= \vec{u}_2 \cdot \vec{v}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|} \cdot (\vec{v}_2^{\parallel} + \vec{z}_2) = 0 + \|\vec{z}_2\| \\ j \geq 2 \qquad r_{rr} &= \vec{u}_j \cdot \vec{v}_j = \frac{\vec{z}_j}{\|\vec{z}_j\|} \cdot (\vec{v}_j^{\parallel} + \vec{z}_j) = \|\vec{z}_j\| \end{aligned}\right\}$$

$$\boxed{\text{Vol} = \det R \text{ !!}}$$

and you can compute $\mathrm{Vol} = \det(R)$ <u>without</u> first doing G.S.!

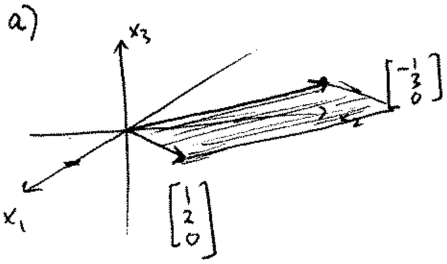
$$A = QR$$
$$A^TA = (QR)^T QR$$
$$= R^T \underbrace{Q^TQ}_{I} R$$
$$A^TA = R^TR$$
$$\det(A^TA) = \det(R^TR) = (\det R^T)(\det R)$$
$$\det(A^TA) = (\det R)^2$$
$$\boxed{\mathrm{Vol} = \sqrt{\det(A^TA)}}$$

(because R, R^T are square matrices – A is <u>not</u> square unless $k=n$)

special case: if $A_{n\times n}$ (n-dim'l parallelepiped in $\mathbb{R}^n$) then $\mathrm{Vol} = \lvert\det(A)\rvert$

<u>examples</u>:

a)



$$A = \begin{bmatrix}1 & -1\\2 & 3\\0 & 0\end{bmatrix}$$

$$A^TA = \begin{bmatrix}1 & 2 & 0\\-1 & 3 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\2 & 3\\0 & 0\end{bmatrix}$$

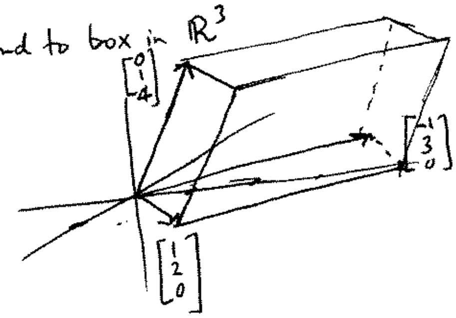
$$= \begin{bmatrix}5 & 5\\5 & 10\end{bmatrix}$$

$\det(A^TA) = 50-25 = 25$

area $= 5$

$= \|\vec{u}\times\vec{v}\|$, by the way. See HW!

c) extend to box in $\mathbb{R}^3$

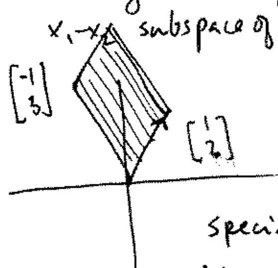


$$A = \begin{bmatrix}1 & -1 & 0\\2 & 3 & 1\\0 & 0 & 4\end{bmatrix}$$

$|A| = 12+8 = 20$

$\boxed{\mathrm{Vol} = 20}$ $= (\text{area of base})(\text{ht}) = 5\cdot 4 = 20$ ✓

b) of course (a) is really happening in x_1-x_2 subspace of $\mathbb{R}^3$:

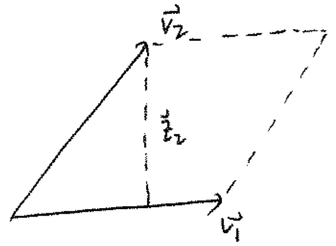


special case

$$\begin{vmatrix}1 & -1\\2 & 3\end{vmatrix} = 3+2 = 5 \ ✓$$

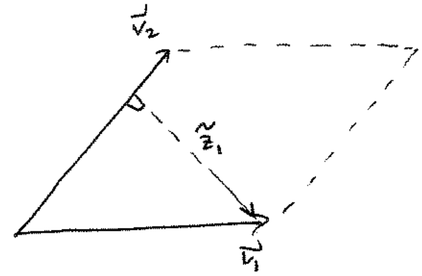
Can you explain why we get the same value for area or volume, no matter what order we list the spanning vectors?

e.g.



(in $\mathbb{R}^n$!)

??



$$A = \|\vec{v}_1\| \, \|\vec{z}_2\| = \sqrt{\det\left(\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}\right)}$$

$$A = \|\vec{v}_2\| \, \|\tilde{z}_1\| = \sqrt{\det\left(\begin{bmatrix} \vec{v}_2^T \\ \vec{v}_1^T \end{bmatrix} \begin{bmatrix} \vec{v}_2 & \vec{v}_1 \end{bmatrix}\right)}$$

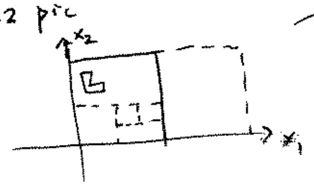
answer :

|Determinant| as area/volume expansion factor
(recall poor Bob)

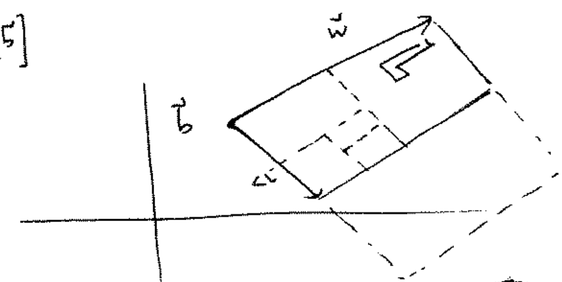
If $T(\vec{x}) = A\vec{x} + \vec{b}$, $T : \mathbb{R}^n \to \mathbb{R}^n$ an affine transformation

Then $|\det(A)|$ = the ratio $\dfrac{\text{area (or vol.) of transformed Bob}}{\text{area (or vol) of original Bob}}$

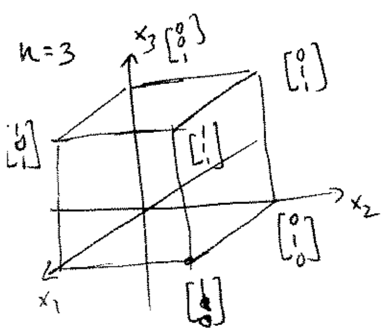
n=2 pic



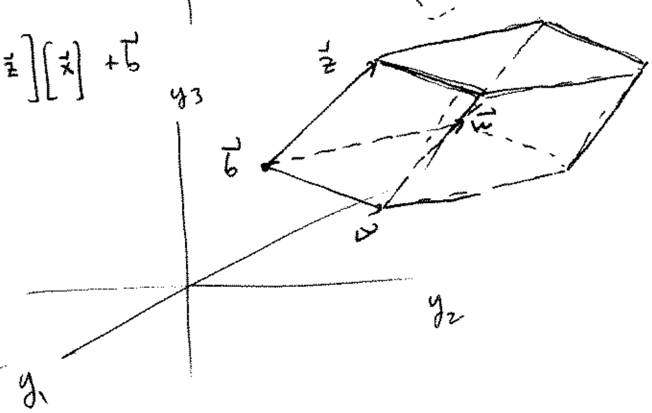
$$T(\vec{x}) = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [5]$$



n=3



$$T(\vec{x}) = \begin{bmatrix} \vec{v} & \vec{w} & \vec{z} \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} + \vec{b}$$



Old
Homework exercises:

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Bob and L-box transform themselves:

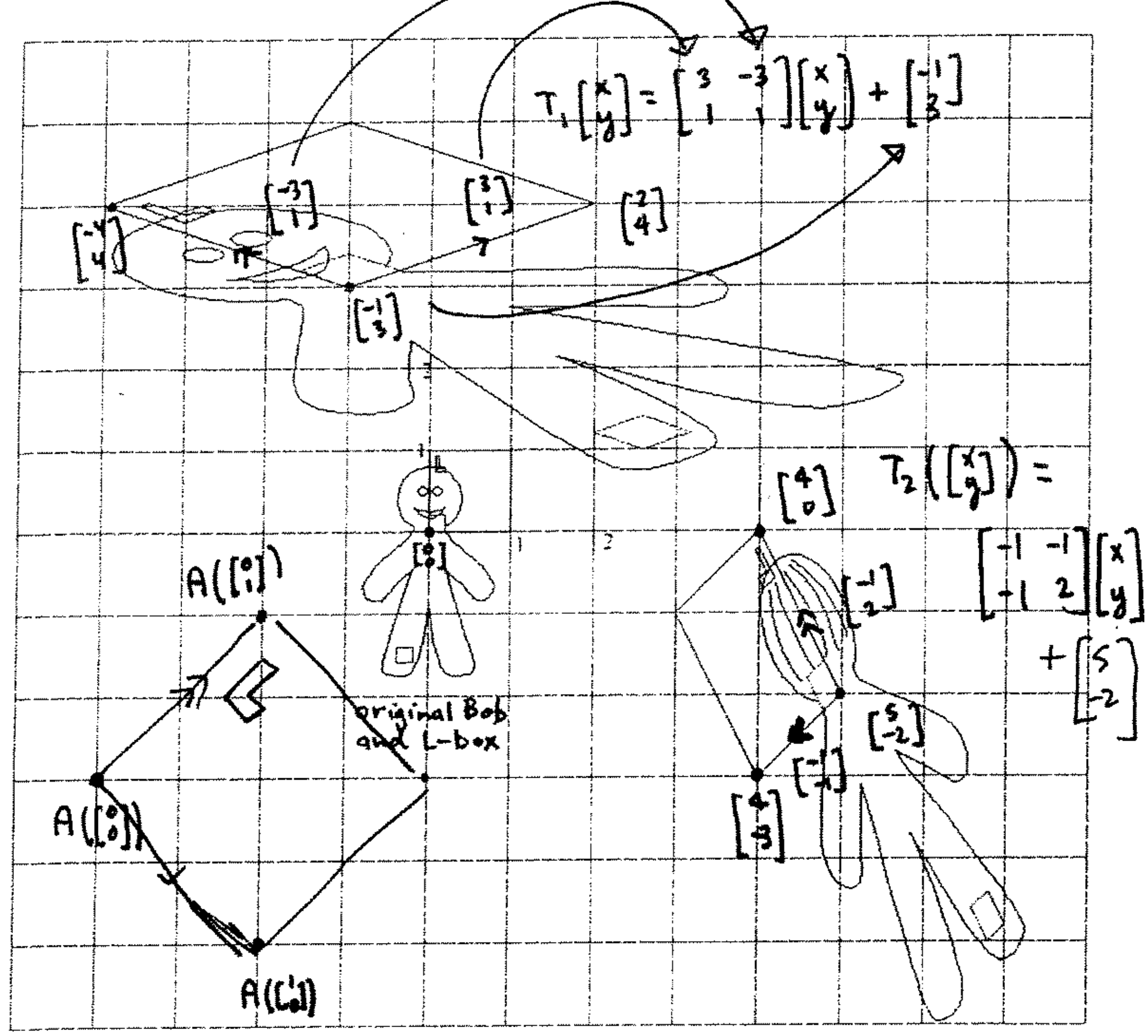
Exercise 4. Find formulas for the two affine maps which are shown.

Exercise 5. Show where the L-box is transformed to by $A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \end{bmatrix}$. If you want, you can draw in a piece of transformed Bob.

$$A\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

$$A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$$A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$



We will use these ideas soon, when we study (iterated function system) fractals.
See http://www.math.utah.edu/~korevaar/fractals

The meaning of whether $\det A > 0$ or $\det A < 0$, in $n \times n$ case:

$A = QR$ $\quad$ Q is $(n\times n)$ orthogonal (provided A is nonsingular)

$|A| = |Q||R|$

$\downarrow$

$\det Q = \pm 1$; the sign of $\det Q$ determines the "orientation" of the columns of A

equivalently, of $\det A$, since $\det R > 0$

$\det Q = +1$, "positively oriented" ("right-handed")

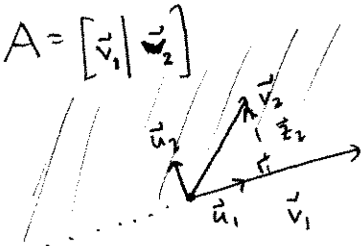
$\det Q = -1$ "negatively oriented" ("left-handed")

any n!

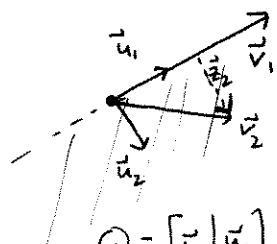
really only makes sense in $\mathbb{R}^2, \mathbb{R}^3$ because that's where our hands live, and we'd also run out of fingers.

e.g. $n=2$:

$A = [\vec{v}_1 | \vec{v}_2]$



vs.



$Q = [\vec{u}_1 | \vec{u}_2] = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$|Q| = +1$

"positively oriented"

$Q = [\vec{u}_1 | \vec{u}_2] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$

"negatively oriented"

similar in $n=3$, $n>3$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\det = +1$ $\longrightarrow$ for $n=3$, if Q is orthogonal and $\det Q = +1$, it actually <u>is</u> a rotation abt. some axis, just like $n=2$.

(for $n>3$, we still <u>call</u> Q a "rotation")

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\det = -1$ $\longrightarrow$ for $n=3$, Q orthog, $|Q| = -1$ then Q is composition of rotation abt an axis, followed by reflection through the orthogonal complement plane of that axis.

In $\mathbb{R}^n$:
What does $\det A > 0$ or $\det A < 0$ mean for the affine transformation $T(\vec{x}) = A\vec{x} + \vec{b}$?