Math 2270-3
Friday Nov. 13

6.2-6.3
~ finish det algebra,
begin geometry

Recall def'n & properties of det.

<u>Exercise 1</u>

$$T(\vec{z}) := \begin{vmatrix} z_1 & 1 & 0 \\ z_2 & 2 & 1 \\ z_3 & -1 & 0 \end{vmatrix}, \quad T: \mathbb{R}^3 \to \mathbb{R} \text{ is linear}$$

What is $\ker T$, $\operatorname{Im}(T)$? (be clever!)
Verify rank + nullity thm.

<u>Exercise 2</u> Let $a \neq b$ be numbers

define $p(t) := \begin{vmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{vmatrix}$

a) Why is $p(t)$ a degree 2 polynomial?

b) What is the coefficient of t^2? Hint: cofactor expand $|A|$ using col. 3.

c) What are two roots of $p(t)$?

d) Why does $p(t) = (b-a)(t-a)(t-b)$?

e) Deduce $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$ 3×3 Vandermonde det.

①

HW for Friday
Nov. 20

6.3 1, 5, 7 8, 11, 13, 18 19, 20, 21,
22, 24, 25 26, 48.
7.1 1, 2, 4, 6, 7, 15-19, 24, 27
30, 31 32 49
7.2 9, 12, 19, 21 25, 26, 27, 28 38

<u>Class exercise</u>

ⓐ Check $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$

$(= \square \text{ area})$

by verifying the algebra:

$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$$

$$\left(= \|\vec{u}\|^2 \|\vec{v}\|^2 \underbrace{(1-\cos^2\theta)}_{\sin^2\theta} \right)$$

ⓑ Check that the area above (call it A) is also computable by the determinant formula

$$A^2 = \det\left(\begin{bmatrix} u^T \\ v^T \end{bmatrix} \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix} \right)$$

which we discuss in class

Exercise 3 Back to exercise 1:

$$T(\vec{z}) = \begin{vmatrix} z_1 & 1 & 0 \\ z_2 & 2 & 1 \\ z_3 & -1 & 0 \end{vmatrix} \quad \text{is linear}$$

Let $\vec{z} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 6 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

What is $T(\vec{z})$? (be clever!)

Exercise 4 Consider the matrix equation

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \vec{x} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}}_{\vec{b}}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the unique solution, so $\vec{b} = x_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

What is $T(\vec{b}) = \begin{vmatrix} \vec{b} & \begin{matrix} 1 \\ 2 \\ -1 \end{matrix} & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \end{vmatrix}$? Deduce $x_1 =$

then similar trick: $x_2 =$

$x_3 =$

Check ans!

If your reasoning in #4 is complete you've just proven Cramer's rule:

Let A be an invertible matrix.

Let $\vec{x}$ solve $A\vec{x} = \vec{b}$

Then

$$x_j = \frac{\left| \mathrm{col}_1(A) \,\vdots\, \cdots \,\vdots\, \vec{b} \,\vdots\, \cdots \,\vdots\, \mathrm{col}_n(A) \right|}{|A|}$$

$\leftarrow$ i.e. numerator is the det of matrix $A_{\vec{b},j}$ obtained from A by replacing its j^{th} column with $\vec{b}$

i.e. $\left| \; col_1(A) \;\vdots\; col_2(A) \;\vdots\; \cdots \;\vdots\; \vec{b} \;\vdots\; col_{j+1}(A) \;\vdots\; \cdots col_n(A) \; \right| = x_j \, |A|$, by linearity of $L_j(\vec{z}) := \det(A_{\vec{z},j})$ and alternating prop of dets!

$\vec{b} = x_1 \, col_1(A) + x_2 \, col_2(A) + \cdots + x_j \, col_j(A) + \cdots + x_n \, col_n(A)$

so, when $|A| \neq 0$, $\boxed{x_j = \dfrac{|A_{\vec{b},j}|}{|A|}. \quad \text{Cramer's rule}}$

<u>Exercise 5</u> : for $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, $cof(A) = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} & +\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 4 \\ 1 & -3 & 6 \end{bmatrix}$

- recall $row_i(A) \cdot row_i(cof(A)) = |A|$
  (expansion down row i)
  $col_j(A) \cdot col_j(cof(A)) = |A|$
  (expand $|A|$ down col_j)

- explain using row substition (& col subs^n),
  why for $i \neq j$
  $row_i(A) \cdot row_j(cof(A)) = 0$
  & $col_i(A) \cdot col_j(cof(A)) = 0$

- Thus $A \cdot cof(A)^T = (\det A) I = cof(A)^T A$

  so since $|A| \neq 0$, $A^{-1} = \dfrac{1}{|A|} cof(A)^T$ ← $cof(A)^T$ is often called the "classical adjoint of A" and written $Adj(A)$

If you've understood exercise completely, you've proven that $(A)(Adj(A)) = (\det A) I = (Adj A) A$ always holds, so that when A^{-1} exists it has the formula

$$\boxed{A^{-1} = \frac{1}{|A|} Adj(A)}$$

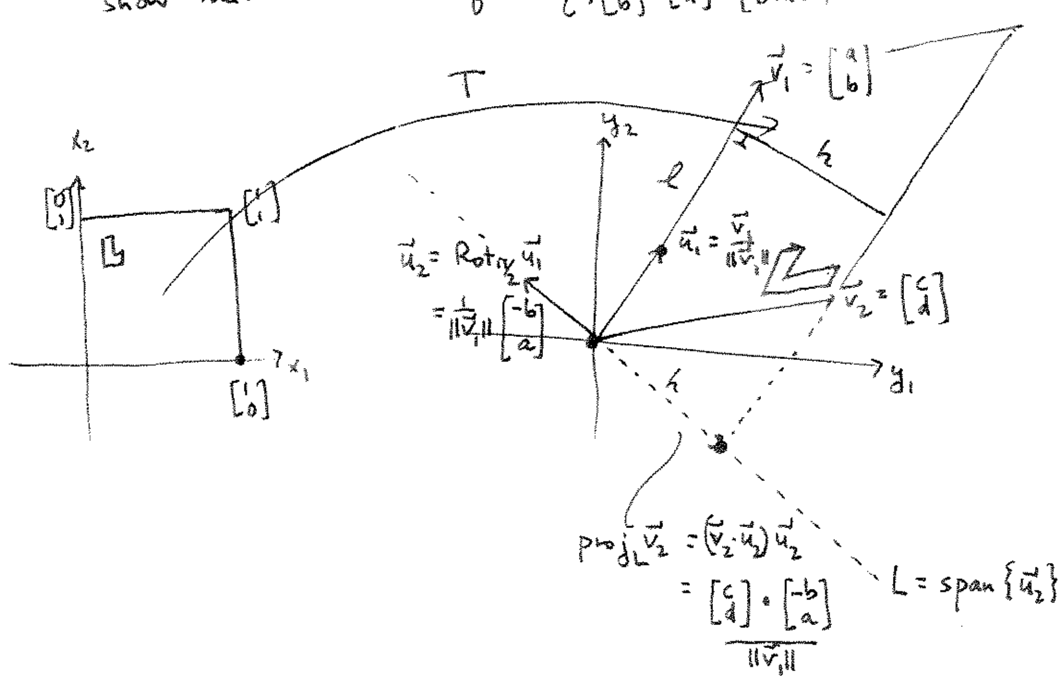
Exercise 6   For $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, check $\frac{1}{|A|}\mathrm{Adj}(A) = A^{-1}$.   Relate to exercise 4.

§ 6.3 Determinant geometry   We'll use the $A = QR$ decomposition to find determinant formulas for volumes of k-dim'l parallelepipeds in $\mathbb{R}^n$ – Monday!
These are the basis for general change of variables formulas in multiple integration, among other applications. ("Jacobian" "parametric integrals")

warmup : Back in September we talked about fact that $|\det(A)|$ was transformed Bob's area expansion factor, and $\det A < 0$ meant $\underbrace{\text{he was flipped over}}_{\text{his "orientation" was changed.}}$, $\det A > 0$ meant he wasn't.

We never gave a general proof. (We did a special case)
Here's one : Because translations don't affect area it suffices to consider the linear map

- $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- Because rect. grids transform to ▱ grids (proportionally) it suffices to show that the area of the $\left\{0, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} a+c \\ b+d \end{bmatrix}\right\}$ ▱ is $|\det A|$, and check signs



Parallelogram
Area $= \ell h$
$= \|\vec{v}_1\|\,\|\vec{v}_2^{\perp}\|$
$= \|\vec{v}_1\|\,\|\mathrm{proj}_L \vec{v}_2\|$
$= \cancel{\|\vec{v}_1\|}\,\frac{|ad-bc|}{\cancel{\|\vec{v}_1\|}}$
$= |\det A|$

(in this example $\det A < 0$, $\vec{v}_2\cdot\vec{u}_2 < 0$)