Math 2270-3

§6.2 the rest of determinant algebra

In Thursday problem session this week we'll carefully go through exam 2, and the ideas/computations it was testing.
(after 15 minutes re Friday hw.)

- $|A| =$
- If A is upper (or lower) Δ'ular, $|A| =$
- $|A|$ can be computed using row or column expansions

From Tuesday notes, check alternating (Theorem 4) and multilinear (Theorem 5) and row-op algebra for det. (Theorem 6)

<u>example</u>

$$\begin{vmatrix} 2 & 0 & 4 \\ 1 & 1 & 1 \\ 3 & 6 & 9 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \\ 3 & 6 & 9 \end{vmatrix} \begin{matrix} R_2 \\ R_1 \\ \end{matrix}$$

$$= -6\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix} \begin{matrix} \\ R_2/2 \\ R_3/3 \end{matrix}$$

$$= -6\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{vmatrix} \begin{matrix} \\ -R_1+R_2 \\ -R_1+R_3 \end{matrix}$$

$$= 6\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix} \begin{matrix} \\ -R_2 \\ \end{matrix}$$

$$= 6\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{vmatrix} \begin{matrix} -R_2+R_1 \\ \\ -R_2+R_3 \end{matrix}$$

$$= 18\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} \\ \\ R_3/3 \end{matrix}$$

$$= 18\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} -2R_3+R_1 \\ R_3+R_2 \\ \end{matrix}$$

$$= 18 \cdot 1 = 18$$

<u>example</u>

$$\begin{vmatrix} 2 & 0 & 4 \\ 1 & 1 & 1 \\ 5 & 3 & 7 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 5 & 3 & 7 \end{vmatrix} \begin{matrix} R_1/2 \\ \\ \end{matrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{vmatrix} \begin{matrix} \\ -R_1+R_2 \\ -5R_1+R_3 \end{matrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} \begin{matrix} \\ \\ -3R_2+R_3 \end{matrix}$$

$$= 2 \cdot 0 = 2$$

<u>Theorem 7!</u> $|A| \neq 0$ iff $\mathrm{rref}(A) = I$ iff A^{-1} exists

<u>because</u>

$$|A| = \underbrace{k_1 k_2 \cdots k_\ell}_{\substack{\text{non-zero factors -} \\ \text{either } (-1)\text{'s from row swaps, or} \\ \text{non-zero factors extracted from} \\ \text{individual rows}}} \underbrace{\det(\mathrm{rref}(A))}_{\neq 0 \text{ iff } \mathrm{rref}(A) = I}$$

Clever consequence of this reasoning:

Theorem 8 $\det(AB) = |A||B|$ (does $\det(A+B) = \det A + \det B$?)

pf:

$$\begin{array}{ccc}
|A| & & |AB| \\
\| & & \| \\
k_1|A_1| & & k_1|A_1B| \\
\| & & \| \\
k_1k_2|A_2| & & k_1k_2|A_2B| \\
\| & & \| \\
\vdots & & \vdots \\
\| & & \| \\
k_1k_2\cdots k_\ell\,|\mathrm{rref}(A)| & & k_1k_2\cdots k_\ell\,|[\mathrm{rref}(A)][B]|
\end{array}$$

doing elementary row ops

do the identical row ops on AB – this is the same as doing them first on A & then multiplying by B!

Case 1 $\mathrm{rref}(A) = I$; $|I| = 1$

thus $|A| = k_1k_2\cdots k_\ell$

so $|AB| = k_1k_2\cdots k_\ell\,|IB|$
$= |A||B|$

Case 2 $\mathrm{rref}(A) \neq I$ In this case

$|A| = 0$ ($\mathrm{rref}(A)$ has zero row)

$|AB| = 0$ ($[\mathrm{rref}(A)][B]$ has zero row too!)

so $0 = |AB| = |A||B|$ holds $\blacksquare$

Theorem 9 How are $|A|$ and $|A^{-1}|$ related?

Theorem 10 If A and B are similar, $B = S^{-1}AS$, how are $|A|, |B|$ related?

Def: Let $T: V \to V$ linear, $\dim V = n$

Let $\mathcal{B} = \{f_1, f_2, \ldots f_n\}$ basis for V, $[T]_{\mathcal{B}} = B$. Define $\det T := |B|$

doesn't depend on choice of basis!

Theorem 11: If Q is an orthogonal matrix $(Q^{-1} = Q^T)$,
what are the possible values of $|Q|$?

recall, only orthogonal Q for $n=2$ are

$$[Rot_\alpha] = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \quad [Refl_{\alpha/2}] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

Theorem 12 Let $A_{n\times n}$ be a square matrix
Let $cof(A)$ be its cofactor matrix, $entry_{ij}(cof(A)) = c_{ij} = (-1)^{i+j}|A_{ij}|$
Defined $adj(A) := cof(A)^T$
Then $\boxed{A\, adj(A) = adj(A)\, A = (\det A)\, I}$
So when A^{-1} exists,
$$\boxed{A^{-1} = \frac{1}{|A|} adj(A)}$$

proof $entry_{ii}(A)(adj(A)) = row_i(A) \cdot col_i(Adj(A))$
$= row_i(A) \cdot row_i(cof(A)) = |A|$ by row expansion for det

$i \neq j$: $entry_{ij}(A)(adj\, A) = row_i(A) \cdot row_j(cof(A))$

$$= \begin{vmatrix} \\ row_i(A) \\ \\ row_i(A) \\ \\ \end{vmatrix} \begin{matrix} \\ \leftarrow \text{row } i \\ \\ \leftarrow \text{row } j \\ \\ \end{matrix}$$

(expand across row_j; the row_j cofactors of this matrix are the row j cofactors of A

$= 0$

(analogous discussion for $(adj\, A)(A) = \det A\, I$)

∎

(we experimented with this formula yesterday)