Math 2270-3
Tuesday Nov. 10

§6.2 determinant algebra

recall for $A_{n\times n}$, concepts of <u>pattern</u> P, <u>inversions</u>, <u>sgn</u>(P), <u>product</u>(P)

$$|A| := \sum_{P} \operatorname{sgn}(P)\operatorname{prod}(P)$$

use this to prove

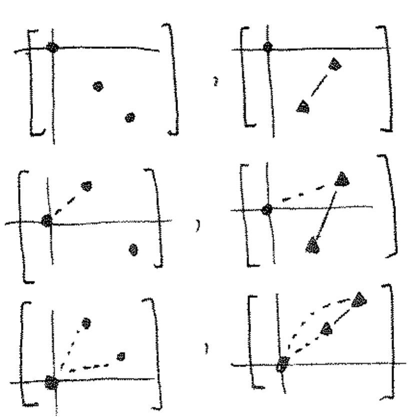
<u>Theorem 2</u> $|A| = |A^T|$ (Monday notes)

<u>Column / row expansions for determinants</u> ("Laplace" or "cofactor" expansion)

<u>Exercise 1</u> Compute $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

by focusing on the first columns & collecting the 6 total patterns by whether they include a_{11}, a_{21}, or a_{31}. Notice how the sgn of the 3×3 pattern is related to the signs of the subsidiary 2×2 patterns:

a_{11}:, a_{21}:, a_{31}:



contribution:

$a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right)$ $= \det(A_{11})$ where A_{11} is 2×2 matrix obtained by deleting row 1 & col 1 of A

$-a_{21}\left(a_{12}a_{33} - a_{32}a_{13}\right)$ $\det(A_{21})$

$+a_{31}\left(a_{12}a_{23} - a_{22}a_{13}\right)$ $\det(A_{31})$

So, we deduced (actually recovered)

$$|A| = a_{11}\det A_{11} - a_{21}\det A_{21} + a_{31}\det A_{31} = col_1(A)\cdot\left[col_2(A)\times col_3(A)\right].$$

Definitions

Let A be $n\times n$, $A=[a_{ij}]$.

The ij-submatrix A_{ij} is the $(n-1)\times(n-1)$ submatrix obtained by deleting row i and col j from A

$\det(A_{ij})$ is the ij-minor, we'll also denote this by m_{ij}

$(-1)^{i+j}\det(A_{ij})$ is the ij-cofactor, also c_{ij}. Write $[c_{ij}] = cof(A)$

Theorem 3 $|A|$ can be computed by expanding across any row, or down any column:

down col j: $$|A| = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij}) = \sum_{i=1}^{n} a_{ij}c_{ij} = col_j(A)\cdot col_j(cof(A))$$

across row i: $$|A| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij}) = \sum_{j=1}^{n} a_{ij}c_{ij} = row_i(A)\cdot row_i(cof(A)).$$

Exercise 2 notice $[-1^{i+j}] = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & \\ - & + & & \end{bmatrix}$ checkerboard. $(-1)^{i+j} = \begin{cases} +1 & i+j \text{ even} \\ -1 & i+j \text{ odd.}\end{cases}$

Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 3 & 0 \\ 2 & 1 & 7 \end{bmatrix}$

a) Find $cof(A)$ matrix

b) check Theorem 3 by taking appropriate dot products.

c) what happens if you dot columns of A with different col's of $cof(A)$?

d) Write down A^{-1} !!

room for cofactor hocus pocus to find A^{-1}

proof of Theorem 3

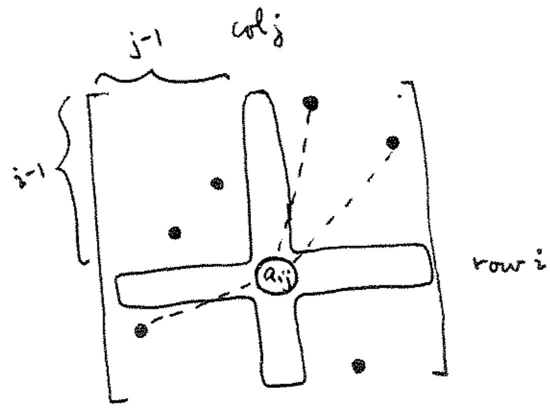
check expansion down col j:

$$|A| := \sum_{P} \operatorname{sgn}(P)\operatorname{prod}(P) \overset{?}{=} \sum_{i=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij} \quad \leftarrow \text{col-expansion}$$

$$= \sum_{i=1}^{n} \left[(-1)^{i+j} a_{ij} \sum_{\substack{\tilde{P} \text{ pattern} \\ \text{in } A_{ij}}} \operatorname{sgn}(\tilde{P}) \operatorname{prod}(\tilde{P}) \right]$$

observe: every pattern P in A corresponds exactly to a choice of one entry a_{ij} in $\mathrm{col}_j(A)$, and a pattern $\tilde{P}$ is submatrix A_{ij}. Therefore the products on the left are exactly the products on the right. The only question is whether the $\pm$'s agree:

If P corresponds to a_{ij}, and $\tilde{P}$ in A_{ij}, is $\boxed{\operatorname{sgn}(P) = (-1)^{i+j} \operatorname{sgn}(\tilde{P})}$?



in this example
$i=5$
$j=4$
$r=2$

let $\tilde{P}$ have
r entries upper left
$\Rightarrow$ $j-1-r$ lower left
$i-1-r$ upper right

✓

$$\text{so } \operatorname{sgn}(P) = \begin{cases} \operatorname{sgn}(\tilde{P}) & i+j \text{ even} \\ -\operatorname{sgn}(\tilde{P}) & i+j \text{ odd.} \end{cases} \Rightarrow P \text{ has } j+i\underbrace{-2-2r}_{\text{even}} \text{ more inversions than } \tilde{P}$$

$4+5-2-4$
$=3$

Alternating and linearity properties of determinant

Theorem 4 If B is obtained from A by swapping 2 rows, then $|B| = -|A|$
(or by swapping 2 cols)
(col results follow from row results: $|A^T| = |A|$)

proof

$n=2$: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad$ ✓

$n=3$: Expand across the row not being swapped. all of the 2×2 determinants in this expansion get their rows swapped, so their signs all change! ∎

thus also for $n = 4, 5, 6, \ldots$

technical proof is to use mathematical induction: If Theorem 4 is true for all $n\times n$ matrices, then consider $(n+1)\times(n+1)$ matrix. Expand across one of the rows which wasn't swapped — all the $n\times n$ dets in expansion are of matrices with swapped rows, so all signs change in expansion ∎

Corollary: If two rows of A are identical (or if two columns are identical), then $|A| = 0$

proof: The B you obtain by swapping those 2 rows, equals A.
Thus $|B| = |A|$, $|B| = -|A|$ so $|A| = 0$.

Theorem 5 (Linearity). Let A be $n\times n$.

a) For $1 \le i \le n$, $T_i(\vec{x}) := \det \underbrace{\begin{bmatrix} row_1(A) \\ row_2(A) \\ \vdots \\ \vec{x}^T \\ \vdots \\ row_n(A) \end{bmatrix}}_{\text{replace } row_i(A) \text{ with } \vec{x}^T} \leftarrow row_i$ is linear

b) For $1 \le j \le n$, $L_j(\vec{x}) = \det \underbrace{\begin{bmatrix} col_1(A) & \cdots & \vec{x} & \cdots & col_n(A) \end{bmatrix}}_{\text{replace } col_j(A) \text{ with } \vec{x}}$ is linear

proof a) notice, replacing $row_i(A)$ with $\vec{x}^T$ does not affect the row_i cofactors $C_{i1}, C_{i2}, \ldots C_{in}$.

Thus $T_i(\vec{x}) = \vec{x}\cdot row_i(cof(A))$

so $T_i(\vec{x}+\vec{y}) = T_i(\vec{x}) + T_i(\vec{y})$

$T_i(k\vec{x}) = kT_i(\vec{x})$

∎

Theorem 6
Application of page 4 to elementary row ops (or elementary column ops)

- swap rows $\longrightarrow$ change sign of det
- $\det \begin{bmatrix} row_1(A) \\ row_2(A) \\ \vdots \\ c\, row_i(A) \\ \vdots \\ row_n(A) \end{bmatrix} = c \det \begin{bmatrix} row_1(A) \\ row_2(A) \\ \vdots \\ row_i(A) \\ \vdots \\ row_n(A) \end{bmatrix}$

  i^{th} row mult'd by c.

  if a constant appears as a multiple for each entry in a row, you can factor it out of det (just part of linearity!)

- $\det \begin{bmatrix} row_1(A) \\ \vdots \\ row_i(A) + c\, row_k(A) \\ \\ \end{bmatrix}$ ← replace row_i with $row_i + c\, row_k$, $k \neq i$

$$= T_i\left(row_i(A) + c\, row_k(A)\right)$$
$$= T_i(row_i(A)) + c\, T_i\, row_k(A)$$
$$= |A| \quad + c \cdot 0$$

this is the det of a matrix with 2 identical rows.

(so, this "hardest" of the 3 elementary row ops leaves det unchanged!)

Exercise 3 recompute $\begin{vmatrix} 1 & -1 & 4 \\ 0 & 3 & 0 \\ 2 & 1 & 7 \end{vmatrix}$ using elementary row ops, elementary column ops, or a combination.