

Name SOLUTIONS
 I.D. number
Math 2270-3
Final Exam
 December 14, 2009

This exam is closed-book and closed-note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and **justify your conclusions**. There are 150 points possible, and the point values for each problem are indicated in the right-hand margin. This exam counts for 30% of your final grade. Good Luck!

1) Let

$$A := \begin{bmatrix} -1 & 2 & 1 & 1 \\ -2 & 4 & 0 & -2 \\ 3 & -6 & 0 & 3 \\ -1 & 2 & 4 & 7 \end{bmatrix}$$

Then $T(x) = Ax$ is a linear map with domain \mathbb{R}^4 and target space \mathbb{R}^4 .

1a) Carefully compute the reduced row echelon form of A . Hint: There should only be two non-zero rows.

(10 points)

$$\begin{array}{l}
 \begin{array}{cccc}
 -1 & 2 & 1 & 1 \\
 -2 & 4 & 0 & -2 \\
 3 & -6 & 0 & 3 \\
 -1 & 2 & 4 & 7
 \end{array} \\
 \hline
 -R_1 \quad \begin{array}{cccc}
 1 & -2 & -1 & -1 \\
 -2 & 4 & 0 & -2 \\
 3 & -6 & 0 & 3 \\
 -1 & 2 & 4 & 7
 \end{array} \\
 \hline
 \begin{array}{cccc}
 1 & -2 & -1 & -1 \\
 0 & 0 & -2 & -4 \\
 0 & 0 & 3 & 6 \\
 0 & 0 & 3 & 6
 \end{array} \\
 \begin{array}{l}
 2R_1 + R_2 \\
 -3R_1 + R_3 \\
 R_1 + R_4
 \end{array} \\
 \hline
 \begin{array}{cccc}
 1 & -2 & -1 & -1 \\
 0 & 0 & 1 & 2 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
 \end{array} \\
 \begin{array}{l}
 R_2 / 2 \\
 -3R_2 + R_3 \\
 -3R_2 + R_4 \\
 R_2 + R_1
 \end{array} \\
 \hline
 \boxed{\begin{array}{cccc}
 1 & -2 & 0 & 1 \\
 0 & 0 & 1 & 2 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
 \end{array}}
 \end{array}$$

$\text{rref}(A)$

1b) Use your work above to answer the following True-False questions about the matrix A in this problem. One point for the correct answer, two points for justification.

(15 points)

1bi) The inverse of A does not exist.

T $\text{rref}(A) \neq I$

1bii) The determinant of A is 147.

F $|A| = 0$

1biii) The nullity of A is 2.

T $\text{rank}(A) = 2$ so $\dim \ker A = 2$

1biv) The columns of A are linearly independent.

F $\ker A \neq \{0\}$. e.g. $\text{col}_2 = -2 \text{col}_1$

1bv) There is no vector b so that the matrix equation $Ax = b$ has exactly one solution x .

T Whenever $Ax = b$ has a sol'n there will be 2 free parameters, (cols in $\text{rref}(A)$ w/o leading 1's) so ∞ 'y many sol's.

1c) Find a basis for the kernel of T .

(5 points)

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}$$

$$\begin{aligned} x_1 &= 2s - t \\ x_2 &= s \\ x_3 &= -2t \\ x_4 &= t \end{aligned}$$

$$\vec{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

basis $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

alternately, since $\dim \ker T = 2$ you could read these column dependencies off of $\text{rref}(A)$ (and/or A). ("Kyle numbers")

1d) Find a basis for the image of T . = span cols of A .

(5 points)

since $\text{col } 2 = -2 \text{ col } 1$ it is redundant & can be removed.

since $\text{col } 4 = \text{col } 1 + 2 \text{ col } 3$ it can be removed. cols 1 & 3 are ind.

$$\Rightarrow \text{a basis is } \{\text{col}_1, \text{col}_3\} = \left\{ \begin{bmatrix} -1 \\ -2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$$

1e) Is the vector $V = \begin{bmatrix} 3 \\ -4 \\ 6 \\ 18 \end{bmatrix}$ in the image of T ?

(5 points)

$$c_1 \begin{bmatrix} -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 6 \\ 18 \end{bmatrix} \Rightarrow 3c_1 = 6 \quad \boxed{c_1 = 2}$$

if $c_1 = 2$ then looking at entry 1:

$$-2 + c_2 = 3 \Rightarrow \boxed{c_2 = 5}$$

$$2 \begin{bmatrix} -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 6 \\ 18 \end{bmatrix} \quad \checkmark \checkmark \checkmark \checkmark$$

Yes!

2a) Find an orthonormal basis of R^3 so that the first two vectors span the plane with implicit equation $-x + y + 2z = 0$, and the third vector is perpendicular to it.

(10 points)

✓
 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for this 2-d plane.
 already \perp ! (else G.S.)
 so just normalize all 3 vects.

It's orthog complement
 is span $\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

$$\begin{array}{ccc} -x & + & y + 2z = 0 \\ -1 & & 1 \quad 2 \end{array}$$

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$\vec{u}_1 \quad \quad \vec{u}_2 \quad \quad \vec{u}_3$

2b) Write the vector $V := \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$ as the sum of a vector in the plane of (2a), with a vector perpendicular to that plane.

$$(\vec{V} \cdot \vec{u}_1) \vec{u}_1 + (\vec{V} \cdot \vec{u}_2) \vec{u}_2$$

$$= \frac{8}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{0}{\sqrt{3}} \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$$

$$\frac{(\vec{V} \cdot \vec{u}_3) \vec{u}_3}{(\vec{V} \cdot \vec{u}_3)}$$

(5 points)

$$\frac{12}{\sqrt{6}} \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}$$

$$\vec{V} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}$$

3) Let W be the subspace of continuous functions with domain \mathbb{R} and image in \mathbb{R} , spanned by the functions $\sin(t)$, $\cos(t)$. Thus a basis for W is

$$\beta := \{\sin(t), \cos(t)\}$$

Let $T: W \rightarrow W$ be defined by

$$T(f) = D^{(2)}(f) + 3D(f) + 2f$$

where D stands for t -derivative. In English, that means that $T(f)$ is the sum of the second derivative of $f(t)$ with three times the first derivative of $f(t)$ with two times the function $f(t)$.

3a) Show that T is linear.

$$T(f) = f'' + 3f' + 2f$$

(5 points)

$$a) T(f+g) = T(f) + T(g):$$

$$\begin{aligned} (f+g)'' + 3(f+g)' + 2(f+g) &= (f'' + 3f' + 2f) + (g'' + 3g' + 2g) \\ &= T(f) + T(g) \quad \checkmark \end{aligned}$$

$$b) \text{ if } k \in \mathbb{R}, T(kf) = kT(f):$$

$$\begin{aligned} (kf)'' + 3(kf)' + 2kf &= kf'' + 3kf' + 2kf \\ &= k(f'' + 3f' + 2f) \\ &= k(T(f)) \quad \checkmark \end{aligned}$$

3b) Find the matrix B for $T: W \rightarrow W$ with respect to the basis β . Recall that the derivative of $\sin(t)$ is $\cos(t)$ and the derivative of $\cos(t)$ is $-\sin(t)$.

$$\beta = \left\{ \underset{\substack{\uparrow \\ f_1}}{\sin t}, \underset{\substack{\uparrow \\ f_2}}{\cos t} \right\}$$

(5 points)

$$[T]_{\beta} = \left[[T(f_1)]_{\beta} \mid [T(f_2)]_{\beta} \right]$$

$$T(\sin t) = -\sin t + 3\cos t + 2\sin t = \sin t + 3\cos t$$

$$[T(f_1)]_{\beta} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$T(\cos t) = -\cos t + 3(-\sin t) + 2\cos t = -3\sin t + \cos t$$

$$[T(f_2)]_{\beta} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\boxed{[T]_{\beta} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}}$$

3c) Suppose an element g of W satisfies

$$T(g(t)) = 10 \cos(t).$$

What matrix equation does the β -coordinate vector of g satisfy? Hint: it involves the matrix you found in 3b!

$$[T(g(t))]_{\beta} = [10 \cos t]_{\beta} \quad (5 \text{ points})$$

$$\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

3d) Solve the matrix equation in (3c), and deduce the solution g to the equation in (3c).

(5 points)

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$g(t) = 3 \sin t + \cos t$$

check

$$\begin{aligned} +2(& g = 3 \sin t + \cos t) \\ +3(& g' = +3 \cos t - \sin t) \\ +1(& g'' = -3 \sin t - \cos t) \end{aligned}$$

$$\begin{aligned} T(g) &= \sin t (6-3-3) \\ &\quad + \cos t (2+9-1) \\ &= 10 \cos t \quad \checkmark \end{aligned}$$

4) This is a discrete dynamical system story problem about an interacting prey-predator population of rabbits and foxes on an isolated island. In this linear model the rabbit and fox populations change each year according to the equations

$$r(t+1) = \frac{5}{2} r(t) - 20 f(t)$$

$$f(t+1) = \frac{1}{10} r(t) - \frac{1}{2} f(t)$$

4a) Explain why this is at least a plausible model, by explaining what the four coefficients imply about how rabbits and foxes would fare in the absence of the other, and about how they interact when both are present.

no foxes: $r(t+1) = \frac{5}{2} r(t)$: rabbits multiply geometrically each year (4 points)

no rabbits: $f(t+1) = -\frac{1}{2} f(t)$ half the foxes die out every year

interactions: $r(t+1)$ is decreased by $20 f(t)$ each year - one fox "causes" 20 rabbit deaths a year

$f(t+1)$ is increase by $\frac{1}{10} r(t)$ each year - each extra 10 rabbits yield one extra fox/year.

4b) Exhibit the transition matrix A for this discrete dynamical system.

(2 points)

$$A = \begin{bmatrix} \frac{5}{2} & -20 \\ \frac{1}{10} & -\frac{1}{2} \end{bmatrix}$$

4c) Find a basis of \mathbb{R}^2 made out of eigenvectors of the transition matrix A . Hint: As a first step, you should find that the eigenvalues are $\frac{1}{2}$ and $\frac{3}{2}$.

(14 points)

$$\begin{vmatrix} \frac{5}{2} - \lambda & -20 \\ \frac{1}{10} & -\frac{1}{2} - \lambda \end{vmatrix} = \lambda^2 - 2\lambda - \frac{5}{4} + 2 = \lambda^2 - 2\lambda - \frac{3}{4} = (\lambda - \frac{1}{2})(\lambda - \frac{3}{2})$$

$$\lambda = \frac{1}{2}$$

$$\begin{array}{cc|c} 2 & -20 & 0 \\ \frac{1}{10} & -1 & 0 \\ \hline 1 & -10 & 0 \\ 0 & 0 & 0 \end{array}$$

$$\vec{v} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

$$\lambda = \frac{3}{2}$$

$$\begin{array}{cc|c} 1 & -20 & 0 \\ \frac{1}{10} & -2 & 0 \\ \hline 1 & -20 & 0 \\ 0 & 0 & 0 \end{array}$$

$$\vec{v} = \begin{bmatrix} 20 \\ 1 \end{bmatrix}$$

$$B = \left\{ \begin{bmatrix} 20 \\ 1 \end{bmatrix}, \begin{bmatrix} 10 \\ 1 \end{bmatrix} \right\}$$

$\lambda = \frac{3}{2} \quad \lambda = \frac{1}{2}$

4d) Suppose that at year zero there are 400 rabbits and 30 foxes. Find an explicit formula for the number of rabbits and foxes "t" years later.

(10 points)

$$\vec{x}(t) = A^t \vec{x}_0$$

$$\vec{x}_0 = \begin{bmatrix} 400 \\ 30 \end{bmatrix} = c_1 \begin{bmatrix} 20 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

$$= 10 \begin{bmatrix} 20 \\ 1 \end{bmatrix} + 20 \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

so $A^t \vec{x}_0 = 10 \left(\frac{3}{2}\right)^t \begin{bmatrix} 20 \\ 1 \end{bmatrix} + 20 \left(\frac{1}{2}\right)^t \begin{bmatrix} 10 \\ 1 \end{bmatrix}$

1	1	30
20	10	400
1	1	30
2	1	40
$-R_1 + R_2$	1	0
R_1	1	1
	1	0
$-R_1 + R_2$	0	1
		20
		$c_1 = 10$
		$c_2 = 20$

4e) For the initial value problem in 4c, what happens to the rabbit and fox populations as $t \rightarrow \infty$?

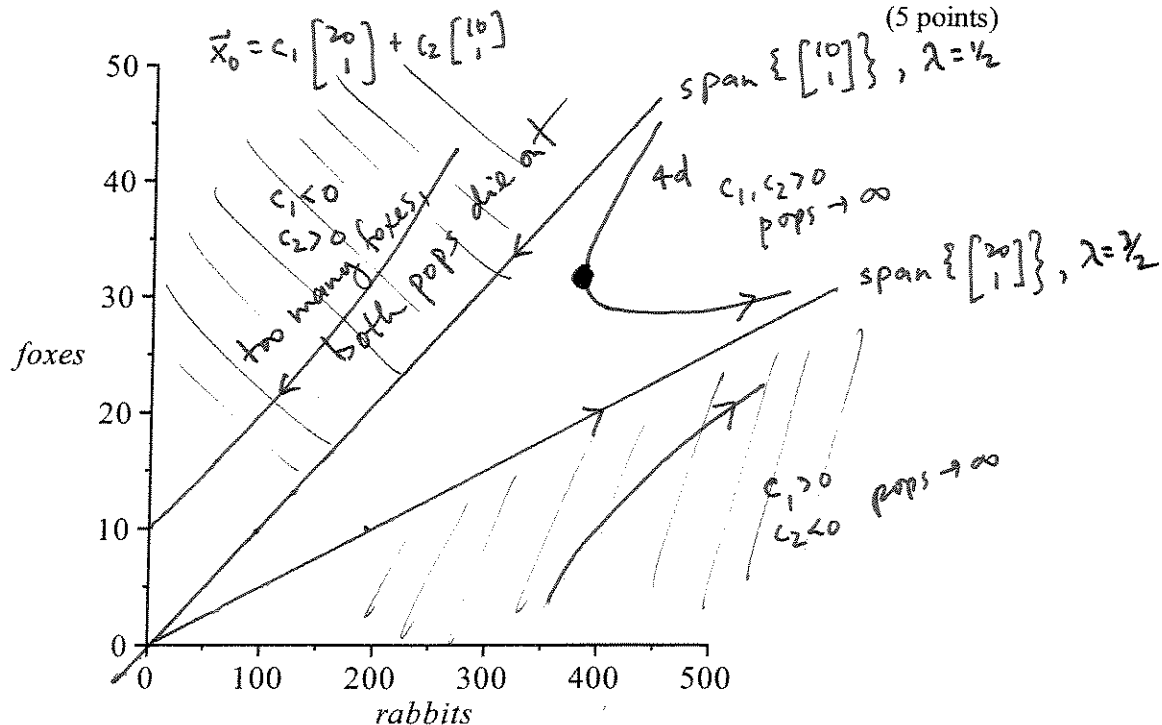
(5 points)

∞ + 0

both pops grow exponentially to ∞ ,
 in an approximate ratio of 20:1

4f) Sketch a phase portrait in the first quadrant of the rabbit-fox plane, and explain what would happen to the rabbit-fox populations depending on different initial populations.

(5 points)



5) Find an orthonormal eigenbasis (of \mathbb{R}^2) for the matrix

$$A := \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

(15 points)

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5)$$

$$\lambda = 0$$

$$\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}$$

$$\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda = 5$$

$$\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

← vectors \perp because λ 's are different
and A is symmetric

normalize

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}.$$

6a) Explain the procedure which allows one to convert a general quadratic equation in n -variables

$$x^T A x + B x + c = 0.$$

into one without any "cross terms". Recall that in the equation above, x represents the standard basis coordinate vector of a point in n -space (written as a column vector), A is a symmetric n by n matrix, B is a row vector (1 by n matrix) and c is a scalar. Be precise in explaining how one finds a new orthonormal basis for R^n so that the standard coordinates of points satisfy the original equation if and only if the coordinates with respect to the new basis satisfy an equation with no cross terms. What theorem about symmetric matrices makes this possible? Show how the new equation is derived from the original one.

$$A \text{ symm} \Rightarrow \exists \text{ o.n. eigenbasis } \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\} \quad (\text{Spectral theorem.}) \quad (4 \text{ points})$$

$$\text{Let } S = [\vec{u}_1 | \vec{u}_2 | \dots | \vec{u}_n] = B$$

$$\text{Write } \vec{x} = S \vec{x}' \quad (\text{Then } \vec{x}' = [\vec{x}]_B)$$

$$\text{so } \vec{x}^T = \vec{x}'^T S^T$$

$$\text{Thus } \vec{x}^T A \vec{x} + B \vec{x} + c = \vec{x}'^T (\underbrace{S^T A S}_{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}) \vec{x}' + (B S) \vec{x}' + c$$

So the quadratic eqn in the new variables is

$$\sum_{i=1}^n \lambda_i (x'_i)^2 + \sum_{i=1}^n b'_i x'_i + c = 0$$

6b) Apply the procedure from part (6a) to find a rotated basis β of R^2 so that the conic section whose standard coordinates satisfy

$$x^2 + 4xy + 4y^2 + 2\sqrt{5}x - \sqrt{5}y + 5 = 0$$

has coordinates $\begin{bmatrix} u \\ v \end{bmatrix}$ with respect to β , and so that the equation these new coordinates satisfy has no $u v$

cross term. Put the resulting equation into standard form and identify the conic. (HINT: Refer to your work in problem 5 !!!)

(8 points)

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2\sqrt{5} & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 5 = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$\uparrow \quad \uparrow$
 $\lambda=5 \quad \lambda=0$

yields

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 2\sqrt{5} & -\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}}_{\begin{bmatrix} 0 & -5 \end{bmatrix}} \begin{bmatrix} u \\ v \end{bmatrix} + 5 = 0$$

$$5u^2 - 5v + 5 = 0$$

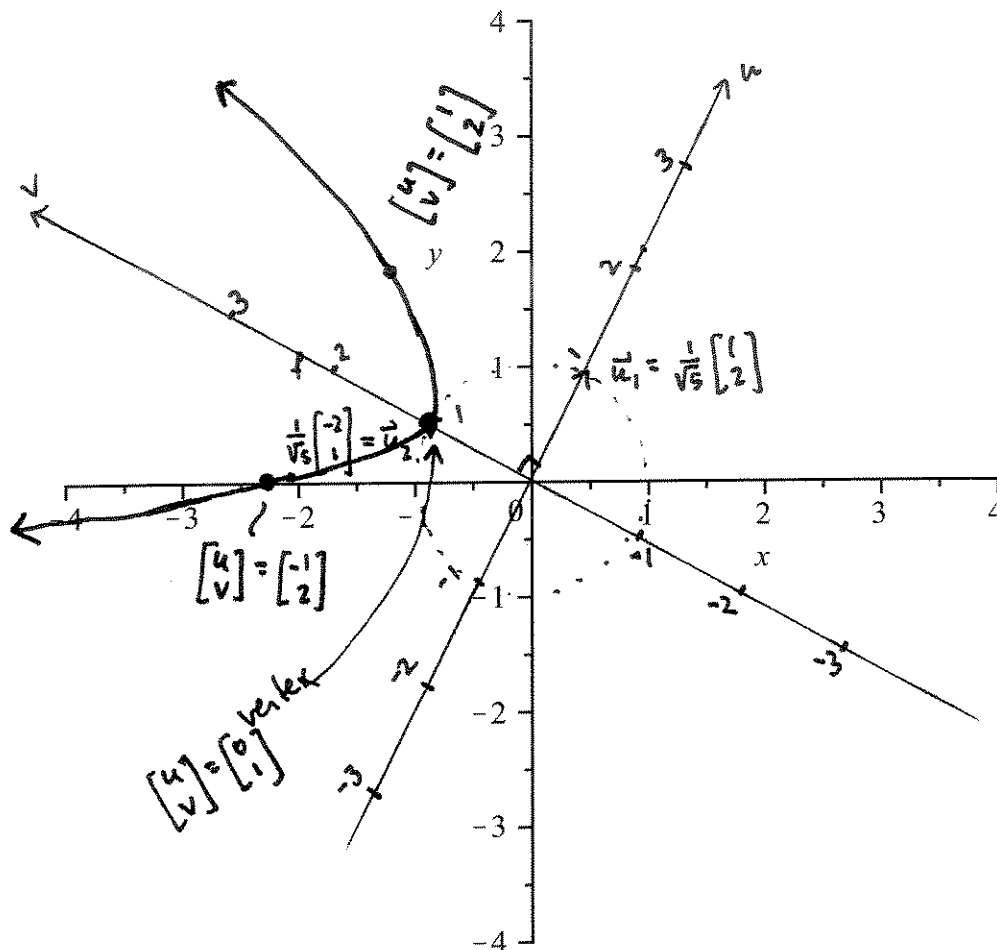
$$5u^2 = 5v - 5$$

$$\boxed{\begin{matrix} u^2 = v - 1 \\ v = u^2 + 1 \end{matrix}} \quad \text{parabola!}$$

6c) Carefully sketch the rotated basis vectors you found in part (6b), the new coordinate axes, and the conic section you found in (6b). Try to measure lengths consistently. Label the vertex or vertices and

exhibit the $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} u \\ v \end{bmatrix}$ coordinates of at least one vertex.

(8 points)



$$\begin{aligned}
 &v = u^2 + 1 \text{ parabola} \\
 &(\text{or } (v-1) = u^2) \\
 &\text{vertex at } \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
 &\text{vertex. } \left\{ \begin{aligned} &S = \begin{bmatrix} x \\ y \end{bmatrix} = S \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \end{aligned} \right.
 \end{aligned}$$