Name: Solutions

I.D. number

Math 2270–3
Exam 2
November 6, 2009

This exam is closed–book and closed–note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and **justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right–hand margin. Good Luck!

1) Consider the plane V in R^3 with basis

$$\beta = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}_{\vec{v}_2} \right\}.$$

1a) Use Gram–Schmidt to find an orthonormal basis for V. (10 points)

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{z}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{3}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}}$$

1b) Find the projection of the vector $\boldsymbol{b} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$ onto V, using your work from (1a). (5 points)

$$\text{proj}_V \vec{b} = (\vec{b}\cdot\vec{u}_1)\vec{u}_1 + (\vec{b}\cdot\vec{u}_2)\vec{u}_2$$

$$= \frac{-6}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{2}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= -2\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}}$$

1c) Find the least squares solution $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ to the system

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

using the "transpose" method.

(10 points)

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{15-9}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} -6 \\ -4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -18 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} -3 \\ 1 \end{bmatrix}}$$

1d) Explain how your solution in (1c) is related to the projection of the vector ***b*** in (1b), and check your claim.

(5 points)

Least squares sol'n solves

$$A\vec{c} = \mathrm{proj}_{\underbrace{\mathrm{Im}(A)}_{= V \text{ in our problem}}} \vec{b}$$

so $\quad A\vec{c} = \mathrm{proj}_V \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$ should hold.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} \checkmark$$

2) We consider the same plane V as in problem (1), with basis

$$\beta = \left\{ \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}} \right\}.$$

2a) Find a basis for the orthogonal complement to V.

(10 points)

$\vec{z} \in V^{\perp}$ if $\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \vec{z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -R_1+R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} -R_2+R_1 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$z_1 = 2t$
$z_2 = -t$
$z_3 = t$

$$\vec{z} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

So basis for $V^{\perp}$ is $\boxed{\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}}$

2b) Exhibit an implicit equation for the plane V, using your work from (2a) (i.e. the plane is the solution set of this equation.)

(5 points)

$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is normal vector for V, so eqtn is

$$\boxed{2x_1 - x_2 + x_3 = 0}$$

2c) Let G (for "good") be the orthonormal basis for R^3 with the property that the first two vectors are the orthonormal basis for V that you found in (1a), and the third vector is a normalized basis vector for the orthogonal complement to V. Display your basis G.

(5 points)

$$G = \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

If $G = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

2d) What is the (simple) matrix B for the projection $T(x) = proj_V(x)$, with respect to your good basis G? Why?

(5 points)

$$B = \left[[T(\vec{u}_1)]_G, [T(\vec{u}_2)]_G, [T(\vec{u}_3)]_G \right]$$

$T(\vec{u}_1) = \vec{u}_1$
$T(\vec{u}_2) = \vec{u}_2$ so $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (projection onto V leaves vectors in V fixed, and $V^\perp$ = kernel.)
$T(\vec{u}_3) = 0$

2e) What is the (more complicated) matrix A for $T(x) = proj_V(x)$, with respect to the standard basis of R^3? You may compute this any way you know how! (If you forgot to memorize a way to get the matrix, you can recover it by expanding the orthonormal basis expression you were using in problem 1 for $proj_V(x)$, so that it ends up being a matrix times x.)

(5 points)

$$A = \left[\vec{u}_1 \middle| \vec{u}_2\right] \begin{bmatrix} \vec{u}_1^T \\ \hline u_2^T \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{5}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

(because $(\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2$

$$= \left[\vec{u}_1 \middle| \vec{u}_2\right] \begin{bmatrix} \vec{x}\cdot\vec{u}_1 \\ \vec{x}\cdot\vec{u}_2 \end{bmatrix} = \left[\vec{u}_1 \middle| \vec{u}_2\right] \begin{bmatrix} \vec{u}_1^T \\ \hline \vec{u}_2 \end{bmatrix} [\vec{x}]$$)

2f) The matrices B and A from (2d) and (2e) are similar. Write down a similarity equation relating them, including an explicit similarity matrix S and its inverse. (If you've done the other problems correctly there's essentially no more computing you need to do for this one, unless you want to check your answer.)

let $E = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ standard basis

(5 points)

$$A = S_{E \leftarrow G} \, B \, S_{G \leftarrow E}$$

Note, $S_{E\leftarrow G} = \left[\vec{u}_1 \middle| \vec{u}_2 \middle| \vec{u}_3\right] = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

so

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}}$$

for "fun", check:

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 5/6 & 1/6 \\ \ldots \end{bmatrix} \checkmark$$

$S_{G\leftarrow E} = S_{E\leftarrow G}^{-1} = S_{E\leftarrow G}^T$ because S is orthogonal

3a) Define what it means for $T: V \to W$ to be *linear*. (3 points)

$\forall f, g \in V, \quad k \in \mathbb{R}$

a) $T(f+g) = T(f) + T(g)$

b) $T(kf) = kT(f)$

3b) Define $kernel(T)$. (3 points)

$= \{ f \in V \text{ s.t. } T(f) = 0 \}$

3c) Define $image(T)$. (3 points)

$= \{ \cancel{\text{is}}\, g \in W \text{ s.t. } g = T(f) \text{ for some } f \in V \}$

3d) Prove that $image(T)$ is a subspace. (6 points)

α) let $g_1, g_2 \in image(T)$

then $g_1 = T(f_1)$, $g_2 = T(f_2)$ some $f_1, f_2 \in V$

so $T(f_1 + f_2) = T(f_1) + T(f_2) = g_1 + g_2$

so $g_1 + g_2 \in image(T)$

β) let $g = T(f) \in image(T)$

then $T(kf) = kT(f) = kg \in image(T)$

thus $image(T)$ is closed under $+$ & scalar mult, so is a subspace of W

4) True–False: 4 points for each problem; two points for the correct answer and two points for the explanation.

(20 points)

4a) If {u,v,w} is any orthonormal collection of vectors, then ||2u+v+2w||=3.

T $\quad (2\vec{u}+\vec{v}+2\vec{w})\cdot(2\vec{u}+\vec{v}+2\vec{w}) = 4\|\vec{u}\|^2+\|\vec{v}\|^2+4\|\vec{w}\|^2 = 9$

because all other dot prods are zero when we expand

4b) There exists an isomorphism from P_3 (the space of polynomials of degree at most 3) to the space of 2x2 matrices.

T $\quad$ Both spaces are isomorphic to $\mathbb{R}^4$, so they are isomorphic to each other

4c) If the columns of a 3x2 matrix A are orthonormal, then

$$[A]\,[A^T] = I$$

F $\quad$ see e.g. the projection matrix in 2e

($A^TA = I_{2\times 2}$ is true, however)

4d) There exists a 2x3 matrix with orthonormal columns.

F $\quad$ orthonormal vectors are independent

there cannot be 3 independent vectors in $\mathbb{R}^2$

4e) If $U = \{f, g\}$ and $B = \{f, f+g\}$ are two bases for a linear space , then the change of basis matrix from U to B is given by

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

T

true!

$$S_{B\leftarrow U} = \left[\, [f]_B \;\middle|\; [g]_B \,\right]$$

$$f = 1f + 0(f+g) \Rightarrow [f]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$g = -1\cdot f + 1\cdot(f+g) \Rightarrow [g]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$