Name...
I.D. number...

Math 2270−3
Exam 2

November 6, 2009

This exam is closed−book and closed−note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and **justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right−hand margin. Good Luck!

1) Consider the plane V in R^3 with basis

$$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

1a) Use Gram−Schmidt to find an orthonormal basis for V.

(10 points)

1b) Find the projection of the vector $\boldsymbol{b} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$ onto V, using your work from (1a).

(5 points)

1c) Find the least squares solution $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ to the system

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

using the "transpose" method.

(10 points)

1d) Explain how your solution in (1c) is related to the projection of the vector $\boldsymbol{b}$ in (1b), and check your claim.

(5 points)

2) We consider the same plane V as in problem (1), with basis

$$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

2a) Find a basis for the orthogonal complement to V .

(10 points)

2b) Exhibit an implicit equation for the plane V, using your work from (2a) (i.e. the plane is the solution set of this equation.)

(5 points)

2c) Let G (for "good") be the orthonormal basis for R^3 with the property that the first two vectors are the orthonormal basis for V that you found in (1a), and the third vector is a normalized basis vector for the orthogonal complement to V. Display your basis G.

(5 points)

2d) What is the (simple) matrix B for the projection $T(x) = proj_V(x)$, with respect to your good basis G? Why?

(5 points)

2e) What is the (more complicated) matrix A for $T(x) = proj_V(x)$, with respect to the standard basis of R^3? You may compute this any way you know how! (If you forgot to memorize a way to get the matrix, you can recover it by expanding the orthonormal basis expression you were using in problem 1 for $proj_V(x)$, so that it ends up being a matrix times x.)

(5 points)

2f) The matrices B and A from (2d) and (2e) are similar. Write down a similarity equation relating them, including an explicit similarity matrix S and its inverse. (If you've done the other problems correctly there's essentially no more computing you need to do for this one, unless you want to check your answer.)

(5 points)

3a) Define what it means for $T : V \to W$ to be *linear.*

(3 points)

3b) Define $kernel(T)$.

(3 points)

3c) Define $image(T)$.

(3 points)

3d) Prove that $image(T)$ is a subspace.

(6 points)

4) True−False: 4 points for each problem; two points for the correct answer and two points for the explanation.

(20 points)

4a) If {u,v,w} is any orthonormal collection of vectors, then ||2u+v+2w||=3.

4b) There exists an isomorphism from P_3 (the space of polynomials of degree at most 3) to the space of 2x2 matrices.

4c) If the columns of a 3x2 matrix A are orthonormal, then

$$[A]\,[A^T] = I.$$

4d) There exists a 2x3 matrix with orthonormal columns.

4e) If $U = \{ f, g \}$ and $B = \{ f, f + g \}$ are two bases for a linear space , then the change of basis matrix from U to B is given by

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$