§8.3 → "optional"; hw won't be graded & we won't cover this section in class, or final exam.

Math 2270-3
Wednesday Dec. 9

- Friday will be review of entire course;
- I'll have review materials posted tonight.
- Saturday (11-1?) will be a chance to go over previous final exams.

§8.2 : positive definite matrices & 2nd derivative test
(this is a rewrite of part of yesterday's notes).

Def : A symmetric matrix A is <u>positive definite</u> iff

$$\vec{x}^T A \vec{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j > 0 \quad \forall \vec{x} \neq \vec{0}.$$

A is <u>negative definite</u> iff $\vec{x}^T A \vec{x} < 0 \quad \forall \vec{x} \neq \vec{0}.$

Theorem: A ← symmetric is positive definite iff all eigenvalues are positive
" negative " " " " negative

proof. Let $S = [\vec{u}_1 | \vec{u}_2 | \cdots \vec{u}_n]$ be an orthogonal matrix of eigenvectors of A.

Consider the usual change of variables formula

$$\vec{x} = S\vec{x}'$$

Then $\vec{x}^T A \vec{x} = \vec{x}'^T S^T A S \vec{x}' = \vec{x}'^T \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{x}' \end{bmatrix} = \sum_{i=1}^{n} \lambda_i (x_i')^2.$

the sum $\sum_{i=1}^{n} \lambda_i (x_i')^2$ is positive $\forall \vec{x}' \neq \vec{0}$ iff all $\lambda_i > 0$
and negative $\forall \vec{x}' \neq \vec{0}$ iff all $\lambda_i < 0$ ∎

A great application of + or − definite matrices is to the 2nd derivative test for functions of several variables.
So let's talk about that, and also look at examples from <u>yesterday's</u> notes

alternate discussion of page 4 Tuesday: Max/Min.

Let $f : \mathbb{R}^n \to \mathbb{R}$

$$f(x_1, x_2, \dots x_n) = f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right)$$

Let $\vec{x}_0 \in \mathbb{R}^n$.

Let $\vec{u} \in \mathbb{R}^n$ a unit vector

Then $\left.\frac{d}{dt}\left(f(\vec{x}_0 + t\vec{u})\right)\right|_{t=0}$ is rate of change of $f$ in the $\vec{u}$ direction, at $\vec{x}_0$.

$$\frac{d}{dt} f(\vec{x}_0 + t\vec{u}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{u}) \overbrace{\frac{d}{dt}(\vec{x}_{0,i} + t u_i)}^{u_i} = \nabla f(\vec{x}_0 + t\vec{u}) \cdot \vec{u}.$$

so @ $t = 0$, this rate of change is $\boxed{\nabla f(\vec{x}_0) \cdot \vec{u}} = D_{\vec{u}} f(\vec{x}_0)$

<u>Definition</u> Let $f$ be a differentiable function

Then $\vec{x}_0$ is a <u>critical point</u> for $f$ iff $\nabla f(\vec{x}_0) = [\frac{\partial f}{\partial x_1}(\vec{x}_0), \frac{\partial f}{\partial x_2}(\vec{x}_0) \dots \frac{\partial f}{\partial x_n}(\vec{x}_0)] = \vec{0}$.

Local extrema of functions occur at critical points, but not all critical points are locations of local extrema.

<u>Def</u> $\left.\frac{d^2}{dt^2} f(\vec{x}_0 + t\vec{u})\right|_{t=0} = D_{\vec{u}\vec{u}} f(\vec{x}_0)$ is the $2^{nd}$ derivative of $f$ in the $\vec{u}$ direction.

$$\frac{d}{dt^2} f(\vec{x}_0 + t\vec{u}) = \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{u}) u_i\right)$$

$$= \sum_{i=1}^{n} u_i \frac{d}{dt}\underbrace{\left(\frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{u})\right)}_{\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{x}_0 + t\vec{u}) \cdot u_j}$$

$$= \sum_{i=1}^{n} u_i u_j \frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{x}_0 + t\vec{u})$$

<u>Hessian matrix</u> $[D^2 f(\vec{x}_0)]$.

@ $t = 0$

$$D_{\vec{u}\vec{u}} f(\vec{x}_0) = \sum_{i=1}^{n} u_i u_j \frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{x}_0) = \vec{u}^T \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0)\right] \vec{u}$$

<u>Therefore</u> $f$ is concave up in every direction $\vec{u}$ at $\vec{x}_0$ iff $[D^2 f(\vec{x}_0)]$ is <u>positive definite</u>

$f$ is concave down " " " " is <u>negative definite</u>

This explains the <u>second derivative test</u> on page 5 Tuesday notes