Math 2270-3
Tuesday Dec. 8
§ 8.1 - 8.2 cont'd. (Tomorrow § 8.3; Friday course review)

- do Example 3 on Monday notes, which is the implicit equation of a quartic surface

Example 4 Identify and graph the solution set to

$$5x^2 + 5y^2 + 6z^2 - 2xy - 4xz - 4yz = 8$$

$$(*) \qquad [x,\ y,\ z]\begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 8$$

$\uparrow$

$G = C + 4I$

$\uparrow$ matrix from example 3: $\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & -2 \\ -2 & -2 & 2 \end{bmatrix}$

so same eigenvects, old evals + 4!

$\lambda = 2$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \lambda = 6$ $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \lambda = 8$ $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

same S: $\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = S \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

$$(**) \qquad [x'\ \ y'\ \ z']\begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = 8$$

finish up! graph!

Spectral Theorem Let $A_{n\times n}$ be a real, symmetric matrix.
Then $\exists$ an orthonormal $\mathbb{R}^n$ basis made of eigenvectors of A, $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_n\}$ $\quad A\vec{u}_j = \lambda_j \vec{u}_j$.
Thus for $S = [\vec{u}_1 | \vec{u}_2 | \cdots \vec{u}_n]$,

$$S^T A S = \begin{bmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & \ddots & \lambda_n \end{bmatrix} \text{ is diagonal.}$$

proof

(1) On Monday we showed that if $\lambda_1 \neq \lambda_2$ are real eigenvalues of A, with eigenvectors $\vec{v}_1, \vec{v}_2 \neq \vec{0}$

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \qquad \text{Then } \vec{v}_1 \perp \vec{v}_2$$
$$A\vec{v}_2 = \lambda_2 \vec{v}_2$$

proof
$$v_2^T A \vec{v}_1 = v_2^T(\lambda_1 \vec{v}_1) = \lambda_1 \vec{v}_2 \cdot \vec{v}_1$$
$$\| $$
$$(v_2^T A^T)\vec{v}_1 = (A\vec{v}_2)^T \vec{v}_1 = \lambda_2 \vec{v}_2^T \vec{v}_1 = \lambda_2 \vec{v}_2 \cdot \vec{v}_1$$
so $\vec{v}_2 \cdot \vec{v}_1 = 0$ ■

We also showed that for $A_{2\times 2}$ symmetric, either A is already diagonal (a multiple of I, in fact), or A has 2 distinct real eigenvalues $\Rightarrow A$ diagonalizable. By (1) the eigenvectors are $\perp$, so normalize to get orthonormal eigenbasis.

(2) All eigenvalues of A are real:
Let $\lambda = a+bi$ be any root of $f_A(\lambda)$, and let $\vec{u}+i\vec{v}$ be a corresponding non-zero eigenvector.

$$A(\vec{u}+i\vec{v}) = (a+ib)(\vec{u}+i\vec{v})$$

take conjugates: $A(\vec{u}-i\vec{v}) = (a-ib)(\vec{u}-i\vec{v})$.

Now consider

$$(\vec{u}-i\vec{v})^T A(\vec{u}+i\vec{v}) = (\vec{u}-i\vec{v})^T(a+ib)(\vec{u}+i\vec{v}) = (a+ib)\left[(\vec{u}-i\vec{v})^T(\vec{u}+i\vec{v})\right]$$
$$= (a+ib)\left[(\vec{u}-i\vec{v})\cdot(\vec{u}+i\vec{v})\right]$$
$$= (a+ib)\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$$

$$\| $$
$$\left((\vec{u}-i\vec{v})^T A^T\right)(\vec{u}+i\vec{v})$$
$$\| $$
$$\left[A(\vec{u}-i\vec{v})\right]^T(\vec{u}+i\vec{v}) = (a-ib)(\vec{u}-i\vec{v})^T(\vec{u}+i\vec{v})$$
$$= (a-ib)\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$$

$\Rightarrow b = 0!$

■

thus $f_A(\lambda)$ factors completely over $\mathbb{R}$.

- If it has n distinct roots, then alg mult = geom mult = 1, all evects w different evals are $\perp$ by (1), and normalize to ~~get~~ orthonormal eigenbasis
- Otherwise it's a little harder: (In practice, if λ_i has alg & geom mult $k_i > 1$ just Gram-~~Stm~~ Schmidt its eigenbasis)

3 General proof, by induction:

Spectral Theorem true for $n=1$ (1×1 matrices <u>are</u> diagonal)

$n=2$ (we checked yesterday).

Inductive step:

- Assume all $(n-1)\times(n-1)$ symmetric matrices are diagonalizable with an orthogonal matrix (with eigenbasis columns).

- Now let $A_{n\times n}$ symmetric

Let λ_1 be any root of $f_A(\lambda)$. λ_1 is real by 2.

Let $\vec{u}_1$ be a unit eigenvector

$$A\vec{u}_1 = \lambda_1\vec{u}_1 \,,\quad \|\vec{u}_1\| = 1.$$

Complete to an orthonormal basis

$$\mathcal{B}_0 = \{\vec{u}_1, \underbrace{\vec{v}_2, \vec{v}_3, \ldots \vec{v}_n}_{\leftarrow \text{ probably not eigenvectors}}\} \text{ of } \mathbb{R}^n$$

$$S_0 = \left[\vec{u}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \ldots \,\vec{v}_n\right] \qquad S_0^T S_0 = I$$

$S_0^T A S_0$ <u>is</u> symmetric (take its transpose!)

1^{st} column <u>is</u> $\begin{bmatrix}\lambda_1\\0\\\vdots\\0\end{bmatrix}$

so $$S_0^T A S_0 = \left[\begin{array}{c|c}\lambda_1 & 0 \\ \hline 0 & \\ 0 & B \\ \vdots & \\ 0 & \end{array}\right]$$

$B_{(n-1)\times(n-1)}$ is symmetric, so by induction hypothesis $\exists$ S_1 orthog,

with $S_1^T B S_1 = \begin{bmatrix}\lambda_2 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$ (λ_i's need not be distinct!)

Thus $$\left[\begin{array}{c|c}1 & 0\\ \hline 0 & S_1^T\end{array}\right]\underbrace{\left[\begin{array}{c|c}\lambda_1 & 0\\ \hline 0 & B\end{array}\right]}_{S_0^T A S_0}\left[\begin{array}{c|c}1 & 0\\ \hline 0 & S_1\end{array}\right] = \begin{bmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{bmatrix}$$

so $S^T A S = D$

$$S = S_0\left[\begin{array}{c|c}1 & 0\\ \hline 0 & S_1\end{array}\right] \quad \text{orthog (product of orthog)}$$

$\blacksquare$

2^{nd} derivative test for multivariable fns

Recall Taylor (McLaurin) for sufficiently differentiable fn $g(t)$ of scalar variable t:

$$\boxed{\bullet \quad g(t) = g(0) + g'(0)\,t + \tfrac{1}{2!} g''(0)\,t^2 + \text{error}}$$

Now let $f : \mathbb{R}^n \to \mathbb{R}$, $f(\vec{x})$

We wish to approximate

$f(\vec{x}_0 + \vec{h})$ for $\vec{h}$ near zero.

write $\vec{h} = t\vec{u}$ $\quad \left(t = \|\vec{h}\|,\ \vec{u} = \frac{\vec{h}}{\|\vec{h}\|}\right)$

define

$g(t) := f(\vec{x}_0 + t\vec{u})$

$\boxed{g(0) = f(\vec{x}_0)}$

to plug into Taylor $\bullet$, need $g'(t), g''(t)$ @ $t=0$.

Chain rule!

$$g'(t) = \frac{d}{dt} f(\vec{x}_0 + t\vec{u}) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{u}) \underbrace{\frac{d}{dt}(\vec{x}_{0,i} + t u_i)}_{u_i}$$

$$= \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{u})$$

so $\boxed{g'(0) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\vec{x}_0)\,u_i = \nabla f(\vec{x}_0) \cdot \vec{u}.}$

$\boxed{t g'(0) = \nabla f(\vec{x}_0) \cdot \vec{h}}$

$$g''(t) = \frac{d}{dt} \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{u})$$

$$= \sum_{i=1}^n u_i \underbrace{\frac{d}{dt}\left(\frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{u})\right)}_{\text{chain rule again}}$$

$$\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{x}_0 + t\vec{u})\,u_j$$

so $\boxed{g''(0) = \sum_{i,j=1}^n u_i u_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0)}$

$$\frac{t^2}{2} g''(0) = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0)\,h_i h_j$$

$$= \frac{1}{2}\vec{h}^T \left[D^2 f(\vec{x}_0)\right] \vec{h}$$

the "Hessian" matrix

entry$_{ij}$ $D^2 f(\vec{x}_0)$

$= \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0)$

Hessian is symmetric!!

$$\boxed{\bullet\bullet \quad f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)\cdot\vec{h} + \tfrac{1}{2}\vec{h}^T\left[D^2 f(\vec{x}_0)\right]\vec{h} + \text{error}}$$

$\vec{x}_0$ is a critical point if $\nabla f(\vec{x}_0) = \vec{0}$

(necessary for local extremum if f is differentiable).

If $\nabla f(\vec{x}_0) = \vec{0}$, then the Hessian term

$$\tfrac{1}{2}\vec{h}^T\left[D^2 f(\vec{x}_0)\right]\vec{h}$$

may determine whether $\vec{x}_0$ is a local extremum (over)

<u>Definition</u>: $A_{n\times n}$ symmetric is <u>positive definite</u> (write $[A]>0$) iff $\vec{h}^T A \vec{h} > 0 \ \forall \vec{h} \neq \vec{0}$

<u>negative definite</u> (write $[A]<0$) iff $\vec{h}^T A \vec{h} < 0 \ \forall \vec{h} \neq \vec{0}$.

<u>Remark</u> Let the columns of S be an orthonormal eigenbasis for A

Let $\vec{h} = S\vec{h}'$ be a change of variables

Then

$$\vec{h}^T A \vec{h}$$
$$= \vec{h}'^T S^T A S \vec{h}'$$
$$= (\vec{h}')^T \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix} \vec{h}' = \sum_{i=1}^{n} \lambda_i (h_i')^2$$

Thus A is pos. def iff all evals >0

A is neg def iff all evals <0

<u>2nd derivative test</u> Let $\vec{x}_0$ be a critical pt of f, $\nabla f(\vec{x}_0) = \vec{0}$. Let f be twice continuously diff'ble at $\vec{x}_0$.

Then

a) $[D^2 f(x_0)] > 0$ (pos def) $\Rightarrow$ $f(x_0)$ local min value

b) $[D^2 f(x_0)] < 0$ (neg def) $\Rightarrow$ $f(x_0)$ local max value

c) if $[D^2 f(x_0)]$ has both positive & negative eigenvalues then $f(x_0)$ is neither a local max or min.

d) $[D^2 f(x_0)] \geq 0$ or ≤ 0 borderline

<u>Examples</u> (these are the ones we've been doing in the past couple lectures!)

① $f(x,y) = 2x^2 + 2y^2 + 5xy$ @ $(0,0)$

$$D^2 f(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$$

$D^2 f(0,0) =$ is twice the matrix in Example 1 (no accident)!

$\lambda_1 = 9$, $\lambda_2 = -1$

case (c)!

2 $g(x,y) = 8x^2 - 16xy + 8y^2$ @ $(0,0)$

should get "borderline"

but you can be clever and deduce $g(0,0)=0$ is a non-strict local min

3 $h(x,y,z) = x^2 + y^2 + 2z^2 - 2xy - 4xz - 4yz$ @ $(0,0,0)$

4 $k(x,y,z) = 5x^2 + 5y^2 + 6z^2 - 2xy - 4xz - 4yz$ @ $(0,0,0)$