Math 2270-3
Monday Dec. 7
§ 8.1-8.2

- We rushed through the example on Friday.
  Let's recap:

Example 1

$$2x^2+2y^2+5xy = [x,y]\underbrace{\begin{bmatrix} 2 & 5/2 \\ 5/2 & 2\end{bmatrix}}_{\text{symmetric.}}\begin{bmatrix} x \\ y\end{bmatrix} = \vec{x}^T A \vec{x}$$

orthonormal eigenbasis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{bmatrix} \quad \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2}\end{bmatrix} \right\}$$

$$\lambda = 9/2 \qquad \lambda = -1/2$$

$$\underbrace{\begin{bmatrix} x \\ y\end{bmatrix}}_{\text{standard coords.}} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\underbrace{\begin{bmatrix} x' \\ y'\end{bmatrix}}_{\text{coords in rotated sys.}}$$

General case

$$\vec{x}^T A \vec{x} = \sum_{i,j=1}^{n} a_{ij}x_i x_j$$

Spectral Theorem $\left\{ \begin{array}{l} A \text{ symmetric} \Rightarrow \\ \exists \text{ orthonormal eigenbasis} \end{array}\right.$

$$\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_n\}$$

$$S := [\vec{u}_1|\vec{u}_2|\cdots \vec{u}_n] = S_{E\leftarrow\mathcal{B}}$$

$$\underbrace{\vec{x}}_{[\vec{x}]_E} = S \underbrace{\vec{x}'}_{[x]_{\mathcal{B}}}$$

so $2x^2+2y^2+5xy$

$$= [x,y]\begin{bmatrix} 2 & 5/2 \\ 5/2 & 2\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix}$$

$$= [x' \ y']\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\underbrace{\begin{bmatrix} 2 & 5/2 \\ 5/2 & 2\end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}}_{\left[\frac{9}{2}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{bmatrix} \ -\frac{1}{2}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2}\end{bmatrix}\right]}\begin{bmatrix} x' \\ y'\end{bmatrix}$$

$$[x' \ y']\begin{bmatrix} 9/2 & 0 \\ 0 & -1/2\end{bmatrix}\begin{bmatrix} x' \\ y'\end{bmatrix}$$

$$= \frac{9}{2}(x')^2 - \frac{1}{2}(y')^2.$$

Now look at graphs $2x^2+2y^2+5xy=1$ & $z = 2x^2+2y^2+5xy$.

so $\vec{x}^T A \vec{x}$

$$= \vec{x}'^T S^T A S \vec{x}' \qquad S^T = S^{-1}$$

$$= \vec{x}'^T D \vec{x}' \qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$$

$$= \lambda_1(x_1')^2 + \lambda_2(x_2')^2 + \cdots + \lambda_n(x_n')^2$$

thus, quadratic expressions in the standard coordinates of $\vec{x}$ can be computed in terms of coordinates of some orthonormal basis so as to eliminate all cross terms.
This is called $\boxed{\text{diagonalizing quadratic forms}}$

Math 2270
Symmetric matrices, quadratic forms, conics and quadrics
Chapter 8

I'm using Maple 12 (not Maple 13), because Maple 13 in the Math Department currently won't draw 3–d plots.

Example 1:

> *with(LinearAlgebra)* :
> *with(plots)* :

> $A := Matrix\left(2, 2, \left[2, \frac{5}{2}, \frac{5}{2}, 2\right]\right)$:
> *Eigenvectors(A)*;
> *Sa* := *Eigenvectors(A)*[2]; *#the second object in the Eigenvectors(A) output is a matrix*

$$\begin{bmatrix} \frac{9}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$Sa := \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad (1)$$

> *basis* := *GramSchmidt*({*Column*(*Sa*, 1), *Column*(*Sa*, 2)}, *normalized* = *true*);
> *#if I don't ask for normalized, Maple doesn't give unit vectors*

$$basis := \left\{ \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \right\} \quad (2)$$

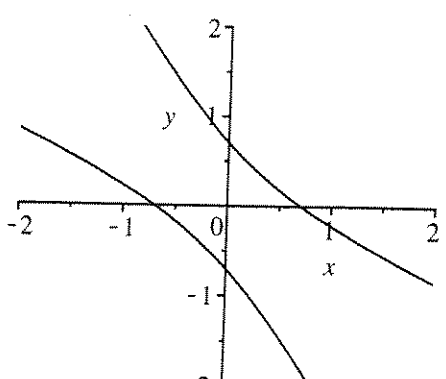
> *S* := (*basis*[2] | *basis*[1]); *#I want positively oriented*

$$S := \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \quad (3)$$

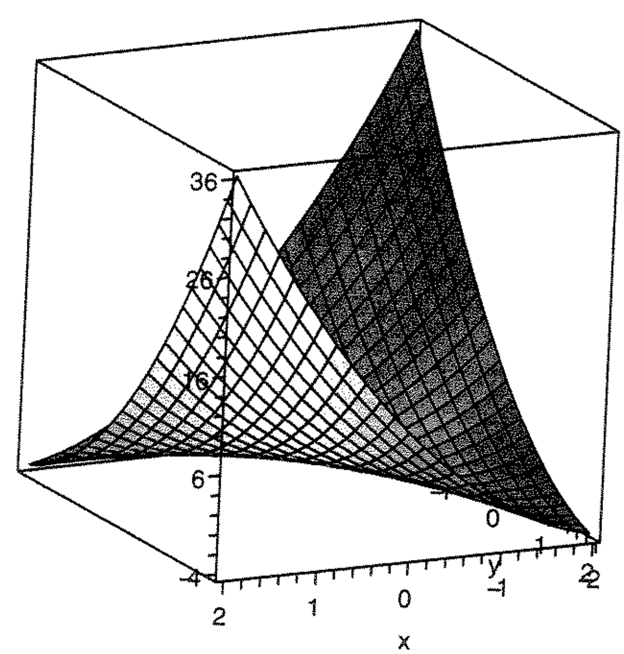
> *Transpose*(*S*).*A*.*S*;

$$\begin{bmatrix} \frac{9}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \quad (4)$$

> $implicitplot(2\cdot x^2 + 2\cdot y^2 + 5\cdot x\cdot y = 1, x = -2..2, y = -2..2, color = black)$;



> $plot3d(2\cdot x^2 + 2\cdot y^2 + 5\cdot x\cdot y, x = -2..2, y = -2..2, axes = boxed)$;



Example 2

Sketch the curve

$$8x^2 - 16xy + 8y^2 + 33\sqrt{2}\,x - 31\sqrt{2}\,y + 70 = 0$$

$$* \quad [x \ \ y]\begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + [33\sqrt{2} \ \ -31\sqrt{2}]\begin{bmatrix} x \\ y \end{bmatrix} + 70 = 0$$

(any time the two diagonal entries of a 2×2 ~~dia~~ symmetric matrix are equal a basis rotated by $\pi/4$ from the standard one will diagonalize it)

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right.$$

$$\lambda = 0 \qquad \lambda = +16$$

subs into $*$: $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$[x' \ \ y']\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\underbrace{\begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ +1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{\left[\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Big| \ 16\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right]}\begin{bmatrix} x' \\ y' \end{bmatrix} + \underbrace{[33\sqrt{2} \ \ -31\sqrt{2}]\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{[2 \ \ -64]}\begin{bmatrix} x' \\ y' \end{bmatrix} + 70 = 0$$

$$\left[\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Big| \ 16\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right]\begin{bmatrix} x' \\ y' \end{bmatrix}$$

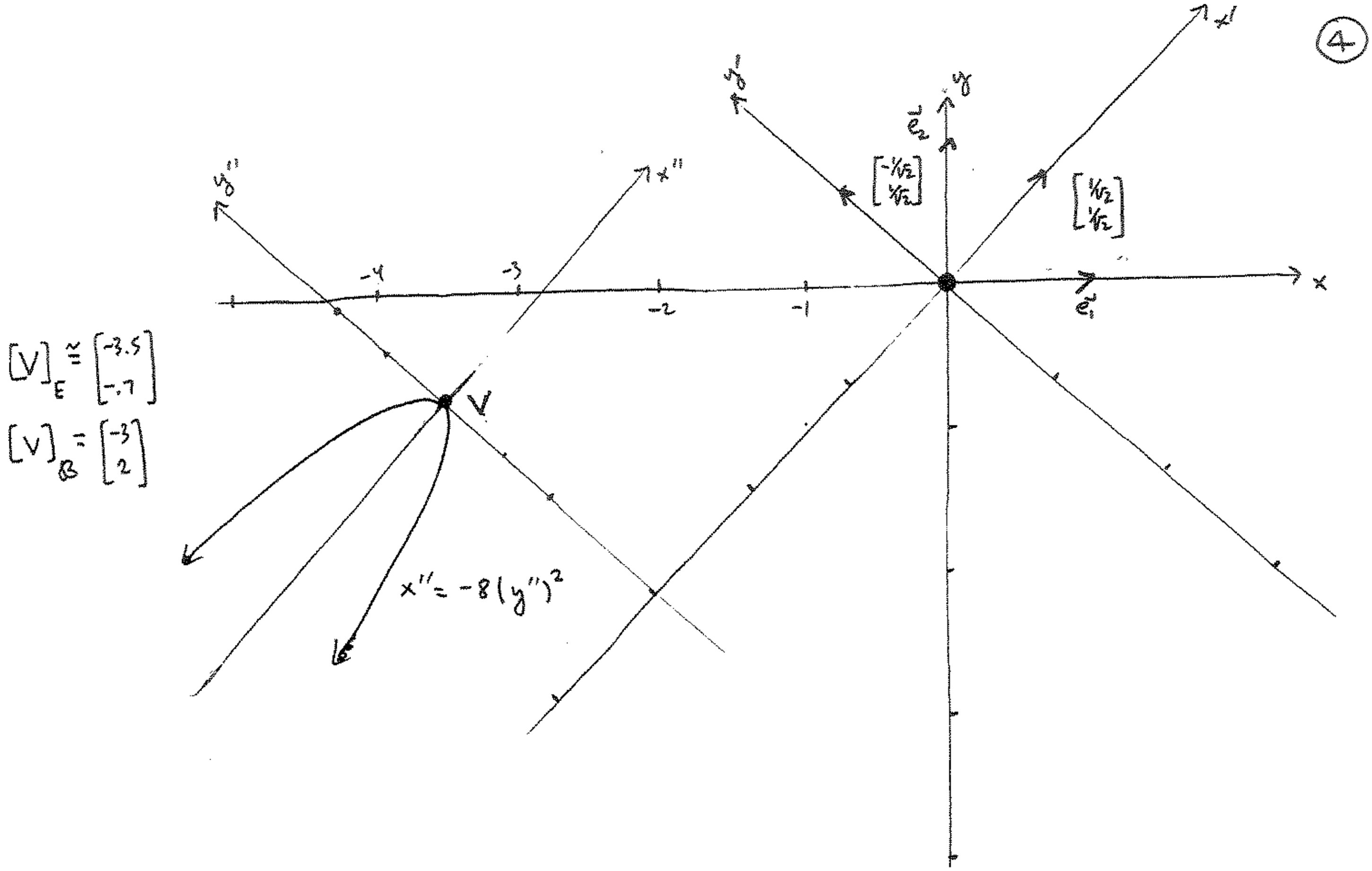
$$[x' \ y']\begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} + [2 \ \ -64]\begin{bmatrix} x' \\ y' \end{bmatrix} + 70 = 0$$

$$16(y')^2 + 2x' - 64y' + 70 = 0$$
$$16((y')^2 - 4y') = -2(x' + 35)$$
$$8((y')^2 - 4y') = -x' - 35$$
$$8(y'-2)^2 - 32 = -x' - 35$$
$$8(y'-2)^2 = -x' - 3$$
$$8(y'-2)^2 = -(x'+3)$$

$$(x'+3) = -8(y'-2)^2$$
$$x'' = -8(y'')^2$$

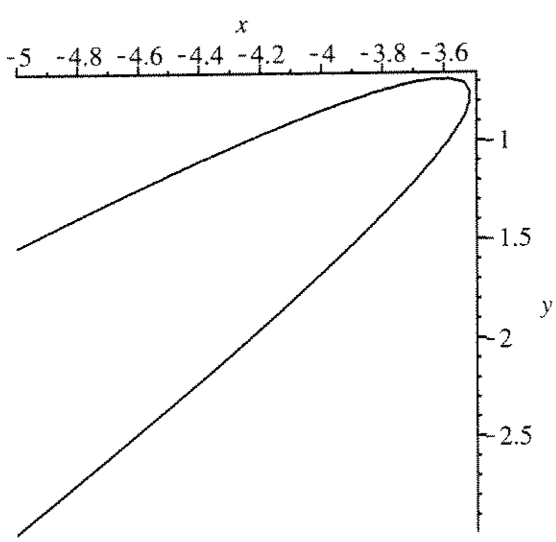
vertex is at $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

so $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} -5/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} -3.5 \\ -.7 \end{bmatrix}$$



Maple check:

```
> with(plots):
  implicitplot(8·x^2 − 16·x·y + 8·y^2 + 33·sqrt(2)·x − 31·sqrt(2)·y + 70 = 0, x = −5 ..0, y = −5
     ..0, color = black, grid = [100, 100]);
```




```
>
```

Example 3 : Identify and graph

$$x^2 + y^2 + 2z^2 - 2xy - 4xz - 4yz = 8$$

(*) $$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & -2 \\ -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 8$$

(Maple) $\lambda = -2$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\lambda = 2$ $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\lambda = 4$ $\begin{bmatrix} -1 \\ -1 \\ +2 \end{bmatrix}$ (pos oriented in this order)

$$S = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

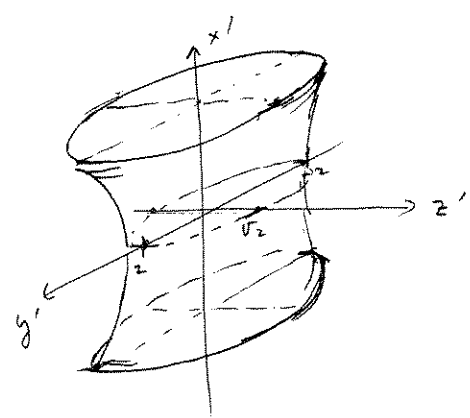
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = S \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

(**) $$\begin{bmatrix} x' & y' & z' \end{bmatrix} \begin{bmatrix} -2 & & 0 \\ & 2 & \\ 0 & & 4 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = 8$$

$$-2(x')^2 + 2(y')^2 + 4(z')^2 = 8$$

$$-(x')^2 + (y')^2 + 2(z')^2 = 4$$

$$(y')^2 + 2(z')^2 = 4 + (x')^2 \leftarrow$$ 1-sheeted hyperboloid opening along x' axis ($\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ direction).



Example 3:

```
> C := Matrix(3, 3, [1,-1,-2,-1, 1,-2,-2,-2, 2]);
```

$$C := \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & -2 \\ -2 & -2 & 2 \end{bmatrix} \tag{9}$$

```
> Eigenvectors(C);
  Sc := Eigenvectors(C)[2]; #the second object in the Eigenvectors(C) output is a matrix
  basis := GramSchmidt({Column(Sc, 1), Column(Sc, 2), Column(Sc, 3)}, normalized = true);
```

$$\begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}$$

$$Sc := \begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

$$basis := \left\{ \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \end{bmatrix} \right\} \tag{10}$$

> *with(plots)* :
implicitplot3d$\left(x^2 + y^2 + 2 \cdot z^2 - 2 \cdot x \cdot y - 4 \cdot x \cdot z - 4 \cdot y \cdot z = 8, x = -3..3, y = -3..3, z = -3..3, axes = boxed\right)$;

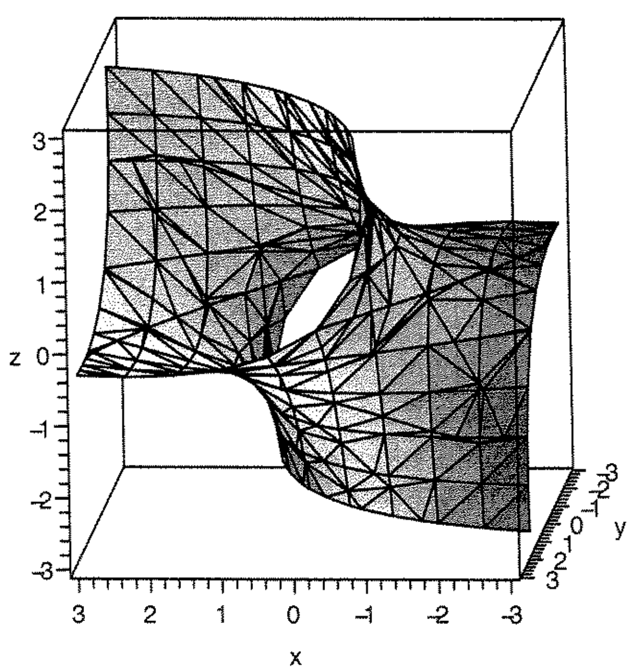


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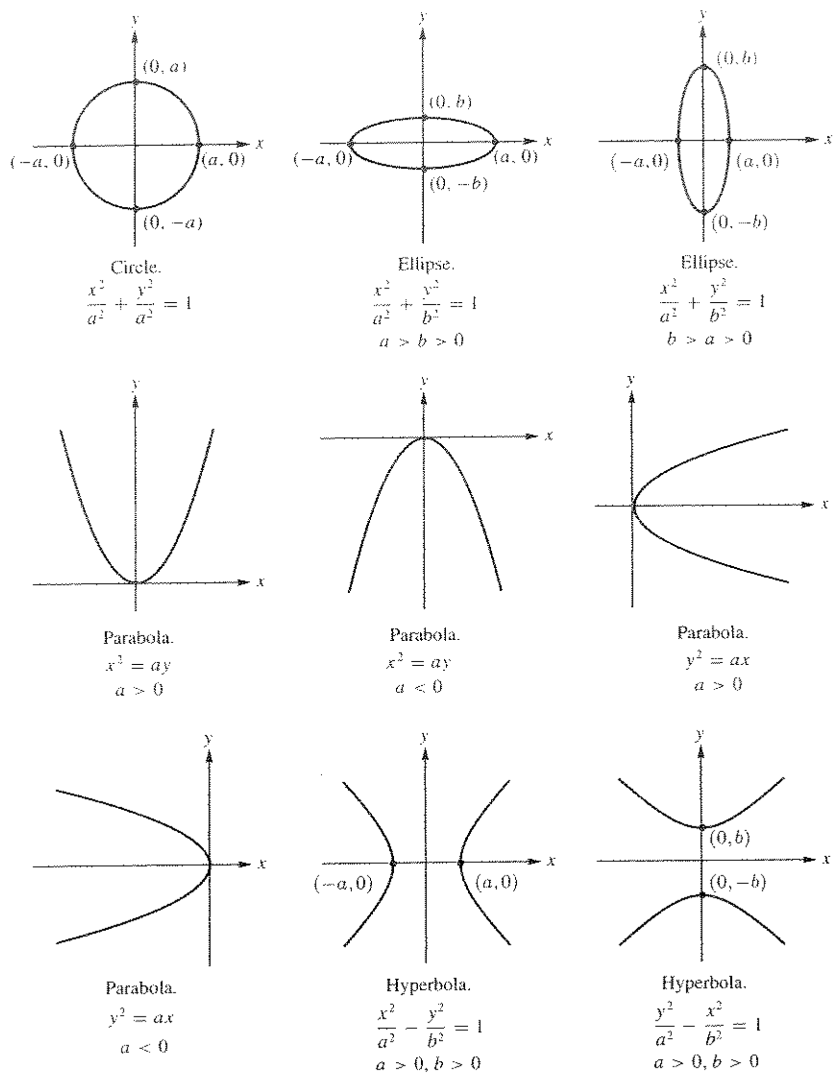
Figure 9.18 ► The conic sections in standard position



Kolman-Hill
"Introductory
Linear Algebra"
7th edition

Solution (a) We rewrite the given equation as

$$\frac{4}{100}x^2 + \frac{25}{100}y^2 = \frac{100}{100}$$

or

$$\frac{x^2}{25} + \frac{y^2}{4} = 1,$$

whose graph is an ellipse in standard position with $a = 5$ and $b = 2$. Thus the x-intercepts are $(5, 0)$ and $(-5, 0)$ and the y-intercepts are $(0, 2)$ and $(0, -2)$.

(b) Rewriting the given equation as

$$\frac{x^2}{9} - \frac{y^2}{4} = 1,$$

we see that its graph is a hyperbola in standard position with $a = 3$ and $b = 2$. The x-intercepts are $(3, 0)$ and $(-3, 0)$.

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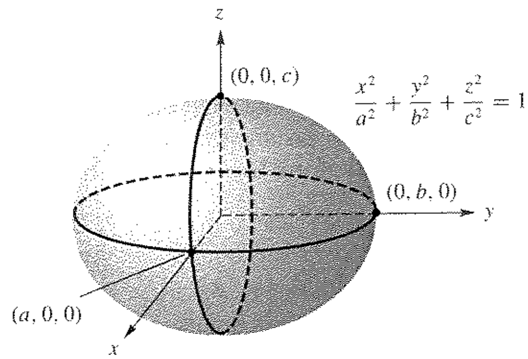

Figure 9.22 ▲
Ellipsoid

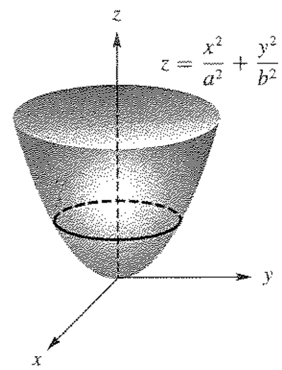

Figure 9.23 ▲
Elliptic paraboloid

A degenerate case of a parabola is a line, so a degenerate case of an elliptic paraboloid is an **elliptic cylinder** (see Figure 9.24), which is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \qquad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

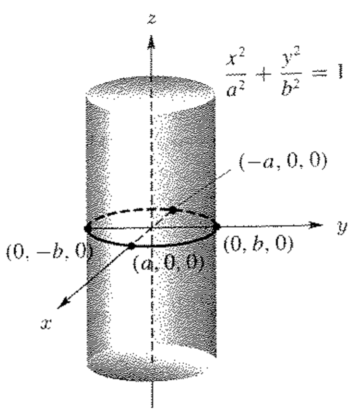

Figure 9.24 ▲
Elliptic cylinder

Hyperboloid of One Sheet (See Figure 9.25.)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

A degenerate case of a hyperboloid is a pair of lines through the origin; hence a degenerate case of a hyperboloid of one sheet is a **cone** (Figure 9.26), which is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0, \qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0.$$

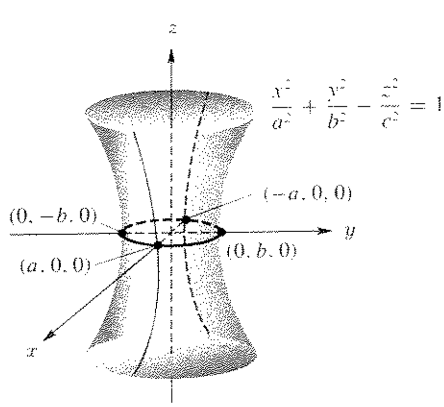

Figure 9.25 ▲
Hyperboloid of one sheet

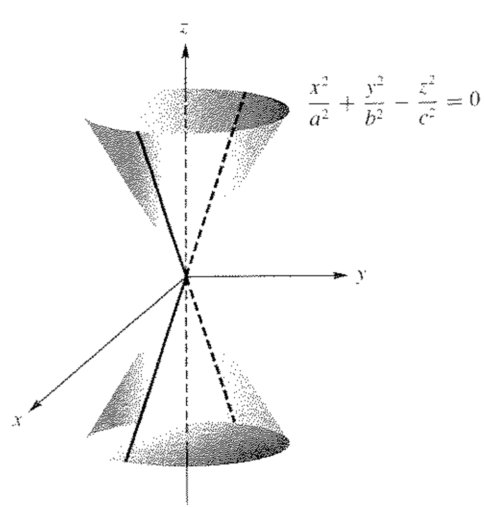

Figure 9.26 ▲
Cone

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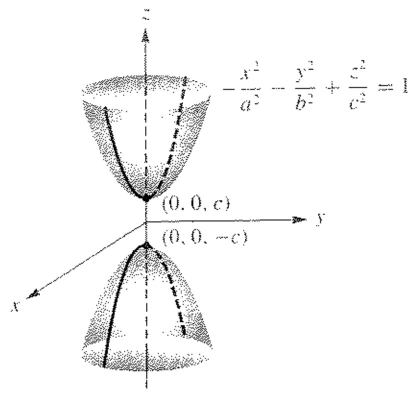

Figure 9.27 ▲
Hyperboloid of two sheets

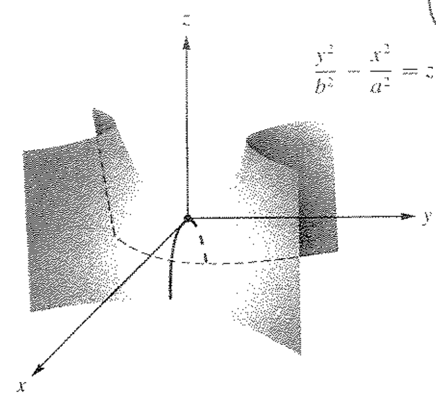

Figure 9.28 ▲
Hyperbolic paraboloid

Hyperboloid of Two Sheets (See Figure 9.27.)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \qquad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Hyperbolic Paraboloid (See Figure 9.28.)

$$\pm z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \qquad \pm y = \frac{x^2}{a^2} - \frac{z^2}{b^2}, \qquad \pm x = \frac{y^2}{a^2} - \frac{z^2}{b^2}.$$

A degenerate case of a parabola is a line, so a degenerate case of a hyperbolic paraboloid is a hyperbolic cylinder (see Figure 9.29), which is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1, \qquad \frac{x^2}{a^2} - \frac{z^2}{b^2} = \pm 1, \qquad \frac{y^2}{a^2} - \frac{z^2}{b^2} = \pm 1.$$

Parabolic Cylinder (See Figure 9.30.) One of a or b is not zero.

$$x^2 = ay + bz, \qquad y^2 = ax + bz, \qquad z^2 = ax + by.$$

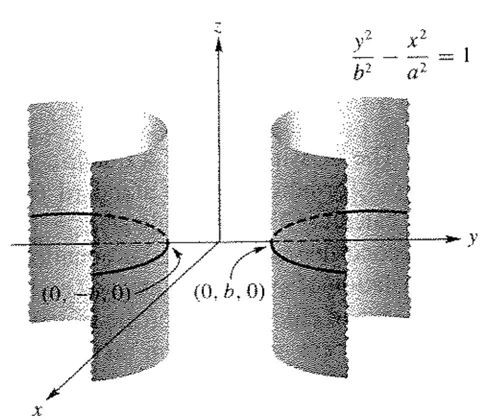

Figure 9.29 ▲
Hyperbolic cylinder

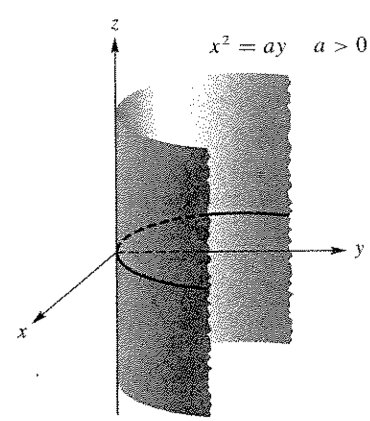

Figure 9.30 ▲
Parabolic cylinder