Math 2270-3
Friday Dec. 4

HW for Fri Dec 11
8.1 (3, 5, 10, 14)
8.2 (15, 16, 19, 20, 21)
8.3 (7, 9, 10, 12)

Last HW! ☺

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- discuss discrete dynamical systems with stochastic or almost stochastic transition matrices, the basis for google page rank, some football rankings etc. (Notes from Wed.)

- Introduction to Chapter 8

with real # entries.

"Spectral theorem"
$\left\{\right.$ All symmetric matrices are diagonalizable, and with n linearly ind. real # eigenvectors (and eigenvalues)

In fact, you can pick your eigen basis to be orthonormal!

Important applications

Example

(1) Identify the curve defined implicitly by $2x^2+2y^2+5xy=1$.

Can you graph it?

(2) Does the function $f(x,y)=2x^2+2y^2+5xy$ have a local max or min at $\begin{bmatrix}0\\0\end{bmatrix}$?

(note, $\nabla f = [f_x, f_y] = [4x+5y,\ 4y+5x] = [0,0]$ at the origin, so $\begin{bmatrix}0\\0\end{bmatrix}$ is a critical point)

Answer
$$2x^2+2y^2+5xy = [x,y]\underbrace{\begin{bmatrix}2 & 5/2\\ 5/2 & 2\end{bmatrix}}_{A,\text{ symmetric}}\begin{bmatrix}x\\y\end{bmatrix}$$

check!

in general, if $\vec{v},\vec{w}$ are column vectors in $\mathbb{R}^n$, and $A_{n\times n}$ then
$$\vec{v}^T A\vec{w} = \sum_{i=1}^{n} v_i\,(\text{entry}_i(A\vec{w}))$$
$$= \sum_{i=1}^{n} v_i\left(\sum_{j=1}^{n} a_{ij}w_j\right)$$
$$= \sum_{i,j=1}^{n} a_{ij}v_iw_j$$

$$|A-\lambda I| = \begin{vmatrix}2-\lambda & 5/2\\ 5/2 & 2-\lambda\end{vmatrix} = (\lambda-2)^2-(5/2)^2$$
$$= (\lambda-2-5/2)(\lambda-2+5/2)$$
$$= (\lambda-9/2)(\lambda+\tfrac{1}{2})$$

$\lambda = 9/2$
$$\left[\begin{array}{cc|c}-\frac{5}{2} & \frac{5}{2} & 0\\ \frac{5}{2} & -\frac{5}{2} & 0\end{array}\right]$$
$\vec{u} = \begin{bmatrix}1\\1\end{bmatrix}$

$\lambda = -1/2$
$$\left[\begin{array}{cc|c}\frac{5}{2} & \frac{5}{2} & 0\\ \frac{5}{2} & \frac{5}{2} & 0\end{array}\right]$$
$\vec{v} = \begin{bmatrix}-1\\1\end{bmatrix}$

Is it an accident that these evects are $\perp$??

This suggests creating an orthonormal eigenbasis!

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\} \qquad S = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad S^{-1} = S^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(I chose it to be positively oriented.)

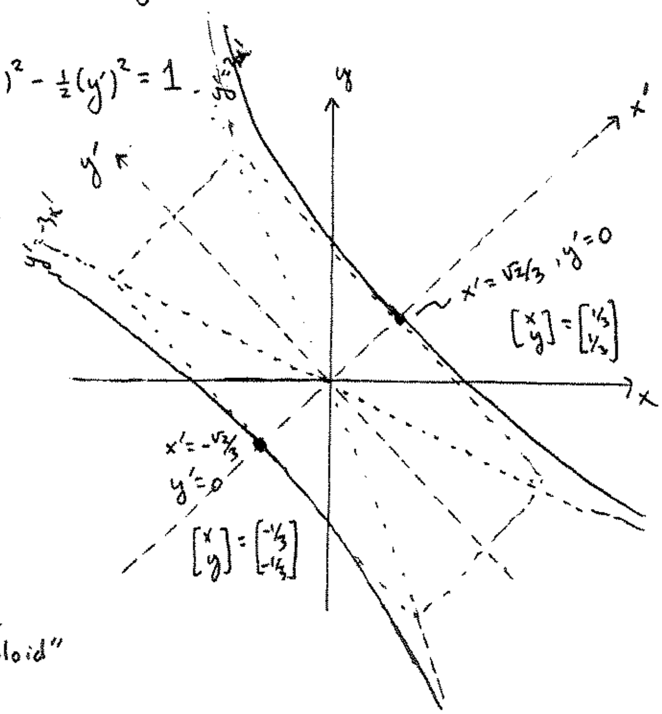
If $\vec{v} \in \mathbb{R}^2$, $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, write $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

then $\begin{bmatrix} x \\ y \end{bmatrix} = S \begin{bmatrix} x' \\ y' \end{bmatrix}$ $\qquad [x, y] = [x', y']\, S^T$ $\qquad$ transpose property!

$$2x^2 + 2y^2 - 5xy = [x, y] \begin{bmatrix} 2 & 5/2 \\ 5/2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= [x' \ y'] \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 5/2 \\ 5/2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad S^{-1} A S = D$$

$$= [x' \ y'] \begin{bmatrix} 9/2 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \tfrac{9}{2}(x')^2 - \tfrac{1}{2}(y')^2 = 1.$$



ans to 1

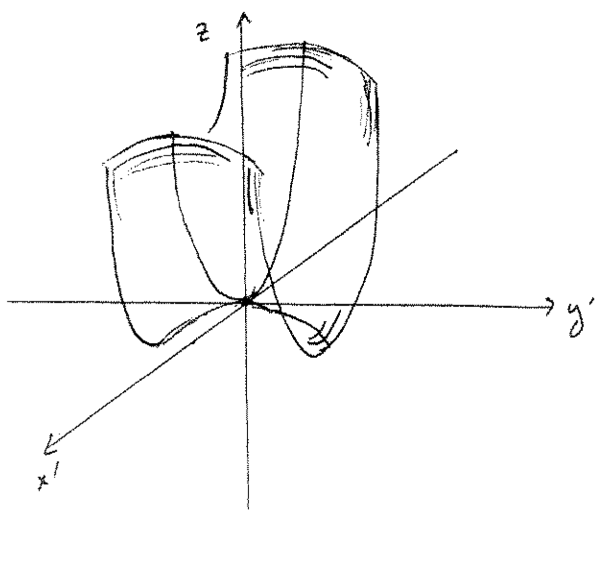
Curve is a hyperbola! $\quad \dfrac{(x')^2}{(\sqrt{2}/3)^2} - \dfrac{(y')^2}{(\sqrt{2})^2} = 1$

NO! not a local min!

ans to 2 $\quad z = 2x^2 + 2y^2 + 5xy$

$z = \frac{9}{2}(x')^2 - \frac{1}{2}(y')^2$ $\qquad$ Saddle surface!

"parabolic hyperboloid"



Big picture:

In order to understand the quadratic form

$$Q(\vec{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \vec{x}^T A \vec{x}$$

A real entries
A chosen to be symmetric

A symm $\Rightarrow$ (by Spectral theorem)

$\exists$ o.n. eigenbasis for $\mathbb{R}^n$, $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, .. \vec{u}_n\}$

so $S = [\vec{u}_1 | \vec{u}_2 | .. \vec{u}_n]$ is an orthog matrix

so $$D = \underbrace{S^T}_{S^{-1}} A S = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 \ddots & 0 \\ 0 & & \lambda_n \end{bmatrix}$$

$$\vec{x} = S[\vec{x}]_{\mathcal{B}} := S\vec{x}'$$

so $$\begin{aligned} Q(\vec{x}) &= \vec{x}^T A \vec{x} \\ &= (\vec{x}')^T S^T A S \vec{x} \\ &= (\vec{x}')^T D \vec{x}' \\ &= \sum \lambda_i (x_i')^2 \end{aligned}$$

applications to conic curves, quartic surfaces, multivariable 2^{nd} deriv test, 2^{nd} order behavior of functions.

Steps in checking spectral theorem:

1. A symmetric $\Rightarrow$ all eigenvalues are real
2. A symmetric $\Rightarrow$ A diagonalizable (over reals)
3. A symmetric $\Rightarrow$ all eigenspaces are mutually orthogonal; i.e. If $A\vec{v} = \lambda_1 \vec{v}$, $A\vec{w} = \lambda_2 \vec{w}$, $\lambda_1 \neq \lambda_2 \Rightarrow \vec{v} \cdot \vec{w} = 0$.
4. A symmetric $\Rightarrow \exists S$ orthog s.t. $S^T A S = D$ diagonal.